A Bishop surface with a vanishing Bishop invariant

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Abstract

We derive a complete set of invariants for a formal Bishop surface near a point of complex tangent with a vanishing Bishop invariant under the action of formal transformations. We prove that the modular space of Bishop surfaces with a vanishing Bishop invariant and with a fixed Moser invariant $s < \infty$ is of infinite dimension. We also prove that the equivalence class of the germ of a generic real analytic Bishop surface near a complex tangent with a vanishing Bishop invariant can not be determined by a finite part of the Taylor expansion of its defining equation. This answers, in the negative, a problem raised by J. Moser in 1985 after his joint work with Webster in 1983 and his own work in 1985. Such a phenomenon is strikingly different from the celebrated theory of Moser-Webster for elliptic Bishop surfaces with non-vanishing Bishop invariants. We also show that a formal map between two real analytic Bishop surfaces with the Bishop invariant $\lambda = 0$ and with the Moser invariant $s \neq \infty$ is convergent. Hence, two real analytic Bishop surfaces with $\lambda = 0$ and $s < \infty$ are holomorphically equivalent if and only if they have the same formal normal form (up to a trivial rotation). Notice that there are many non-convergent formal transformations between Bishop surfaces with $\lambda = 0$ and $s = \infty$. Notice also that a generic formal map between two real analytic hyperbolic Bishop surfaces is divergent as shown by Moser-Webster and Gong. Hence, Bishop surfaces with a vanishing Bishop invariant and $s \neq \infty$ behave very differently, in this respect, from hyperbolic Bishop surfaces or elliptic Bishop surfaces with $\lambda = 0$ and $s = \infty$. We also show that a Bishop surface with $\lambda = 0$ and $s < \infty$ generically has a trivial automorphism group and has the largest possible automorphism group if and only if it is biholomorphic to the model surface $M_s = \{(z, w) \in \mathbb{C}^2 : \ w = |z|^2 + z^s + \overline{z}^s\}$. Notice that, by the Moser-Webster theorem, an elliptic Bishop surface with $\lambda \neq 0$, always has automorphic group $\mathcal{Z}_2$. Hence, Bishop surfaces with $\lambda = 0$ and $s \neq \infty$ have the similar character as that of strongly pseudoconvex real hypersurfaces in the complex spaces of higher dimensions.

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1 Introduction and statements of main results

In this paper, we study the precise holomorphic structure of a real analytic Bishop surface near a complex tangent point with the Bishop invariant vanishing. A Bishop surface is a generically embedded real surface in the complex space of dimension two. Points on a Bishop surface are either totally real or have non-degenerate complex tangents. The holomorphic structure near a totally real point is trivial. Near a point with a complex tangent, namely, a point with a non-trivial complex tangent space of type $(1, 0)$, the consideration could be much more subtle. The study of this problem was initiated by the celebrated paper of Bishop in 1965 [Bis], where for a point p on a Bishop surface M with a complex tangent, he defined an invariant λ now called the Bishop invariant. Bishop showed that there is a holomorphic change of variables, that maps p to 0, such that M, near $p = 0$, is defined in the complex coordinates $(z, w) \in \mathbb{C}^2$ by

$$w = z\overline{z} + \lambda(z^2 + \overline{z}^2) + o(|z|^2), \tag{1.1}$$

where $\lambda \in [0, \infty]$. When $\lambda = \infty$, (1.1) is understood as $w = z^2 + \overline{z}^2 + o(|z|^2)$. It is now a standard terminology to call p an elliptic, hyperbolic or parabolic point of M, according to whether $\lambda \in [0, 1/2)$, $\lambda \in (1/2, \infty)$ or $\lambda = 1/2, \infty$, respectively.

Bishop discovered an important geometry associated with M near an elliptic complex tangent p by proving the existence of a family of holomorphic disks attached to M shrinking down to p. He also proposed several problems concerning the uniqueness and regularity of the geometric object obtained by taking the union of all locally attached holomorphic disks. These problems, including their higher dimensional cases, were completely answered through the combining efforts of many people. (See [KW1], [BG], [KW2], [MW], [Mos], [HK], [Hu3]; in particular, see [KW1], [MW], [Hu3]).

Bishop invariant is a quadratic invariant, capturing the basic geometric character of the surface. The celebrated work of Moser-Webster [MW] first investigated the more subtle higher order invariants. Different from Bishop's approach of using the attached holomorphic disks, Moser-Webster's starting point is the existence of a more dynamically oriented object: an intrinsic pair of involutions on the complexification of the surface near a non-exceptional complex tangent. Here, recall that the Bishop invariant is said to be non-exceptional if $\lambda \neq 0, 1/2, \infty$ or if $\lambda\nu^2 - \nu + \lambda = 0$ has no roots of unity in the variable ν. Moser-Webster proved that, near a non-exceptional complex tangent, M can always be mapped, at least, by a formal transformation to the normal form defined in the complex coordinates $(z, w = u + iv) \in \mathbb{C}^2$ by:

$$u = z\overline{z} + (\lambda + \epsilon u^s)(z^2 + \overline{z}^2)\ ,\ v = 0\ ,\ \epsilon \in \{0, 1, -1\}\ ,\ s \in \mathbb{Z}^+. \tag{1.2}$$

Moser-Webster also provided a convergence proof of the above mentioned formal transformation in the non-exceptional elliptic case: $0 < \lambda < 1/2$. However, the intriguing elliptic case with $\lambda = 0$ has to be excluded from their theory. Instead, Moser in [Mos] carried out a study for $\lambda = 0$ from a more formal power series point of view. Moser derived the following formal

pseudo-normal form for M with $\lambda = 0$:

$$w = z\overline{z} + z^s + \overline{z}^s + 2Re\{\sum_{j\geq s+1} a_j z^j\}. \tag{1.3}$$

Here s is the simplest higher order invariant of M at a complex tangent with a vanishing Bishop invariant, which we call the Moser invariant. Moser showed that when $s = \infty$, M is then holomorphically equivalent to the quadric $M_\infty = \{(z, w) \in \mathbb{C}^2 : w = |z|^2\}$.

Moser's formal pseudo-normal form is still subject to the simplification of a very complicated infinitely dimensional group $aut_0(M_\infty)$, the formal self-transformation group of M_∞. And it was left open from the work of Moser [Mos] to derive any higher order invariant other than s from the Moser pseudo-normal form. At this point, we mention that $aut_0(M_\infty)$ contains many non-convergent elements. Based on this, Moser asked two basic problems concerning a Bishop surface near a vanishing Bishop invariant in his paper [Mos]. The first one is concerning the analyticity of the geometric object formed by the attached disks up to the complex tangent point. This was answered in the affirmative in [HK]. Hence, the work of [HK], together with that of Moser-Webster [MW], shows that, as far as the analyticity of the local hull of holomorphy is concerned, all elliptic Bishop surfaces are of the same character. The second problem that Moser asked is concerning the higher order invariants. Notice that by the Moser-Webster normal form, an analytic elliptic Bishop surface with $\lambda \neq 0$ is holomorphically equivalent to an algebraic one and possesses at most two more higher order invariants. Moser asked if M with $\lambda = 0$ is of the same character as that for elliptic surfaces with $\lambda \neq 0$. Is the equivalence class of a Bishop surface with $\lambda = 0$ determined by an algebraic surface obtained by truncating the Taylor expansion of its defining equation at a sufficiently higher order level? Gong showed in [Gon2] that under the equivalence relation of a smaller class of transformation group, called the group of holomorphic symplectic transformations, M with $\lambda = 0$ does have an infinite set of invariants. However, under this equivalence relation, elliptic surfaces with non-vanishing invariants also have infinitely many invariants. Gong's work later on (see, for example, [Gon2-3] [AG]) demonstrates that as far as many dynamical properties are concerned, exceptional or non-exceptional hyperbolic, or even parabolic complex tangents are not much different from each other.

In this paper, we derive a formal normal form for a Bishop surface near a vanishing Bishop invariant, by introducing a quite different weighting system. This new weighting system fits extremely well in our setting and may have applications in many other problems. We will obtain a complete set of invariants under the action of the formal transformation group. We show, in particular, that the modular space for Bishop surfaces with a vanishing Bishop invariant and with a fixed (finite) Moser invariant s is an infinitely dimensional manifold in a Frèchet space. This then immediately provides an answer, in the negative, to Moser's problem concerning the determination of a Bishop surface with a vanishing Bishop invariant from a finite truncation of its Taylor expansion. Furthermore, it can also be combined with some already known arguments to show that most Bishop surfaces with $\lambda = 0$, $s \neq \infty$ are not holomorphically equivalent to algebraic surfaces. Hence, one sees a striking difference of an elliptic Bishop surface with a

vanishing Bishop invariant from elliptic Bishop surfaces with non-vanishing Bishop invariants. The general phenomenon that the infinite dimensionality of the modular space has the consequence that any subclass formed by a countable union of finite dimensional spaces is of the first category in the modular space seems already clear even to Poincaré [Po]. In the CR geometry category, we refer the reader to a paper of Forstneric [For] in which the infinite dimensionality of the modular space of generic CR manifolds is used to show that CR manifolds holomorphically equivalent to algebraic ones form a very thin set among all real analytic CR manifolds. Similar to what Forstneric did in [For], our argument to show the generic non-algebraicity from the infinite dimensionality of the modular space also uses the Baire category theorem.

It is not clear to us if the new normal form obtained in this paper for a real analytic Bishop surface with $\lambda = 0,\ s < \infty$ is always convergent. However, we will show that if the formal normal form is convergent, then the map transforming the surface to its normal form must be convergent in case the Moser invariant $s \neq \infty$. Remark that there are many non-convergent formal maps transforming real analytic Bishop surfaces with a vanishing Bishop invariant and with $s = \infty$ to the model surface M_∞ defined before. (See [MW] [Mos] [Hu2]). Hence, our convergence theorem reveals a non-trivial role that the Moser invariant has played in the study of the precise holomorphic structure of a Bishop surface with $\lambda = 0$. At this point, we would like to mention that there are many other different problems where one also considers the convergence of formal power series, though very different methods and approaches need to be employed in different settings. To name a few, we here mention the papers of Baouendi-Ebenfelt-Rothschild [BER][BMR][MMZ], Webster [We], Stolovitch [St] and the references therein. In the research described in [BER][BMR][MMZ], one tries to understand the convergence of formal CR maps between not too degenerate real analytic CR manifolds. In [We] [Sto], one encounters other type of convergence problems in the normalization of real submanifolds in $\mathbb{C}^n$.

Our convergence argument uses the Moser-Webster [MW] polarization, as in the non-vanishing Bishop invariant case treated by Moser-Webster. However, different from the Moser-Webster situation, we do not have a pair of involutions, which were the starting point of the Moser-Wbetser theory. The main idea in the present paper for dealing with our convergence problem is to find a new surface hyperbolic geometry, by making use of the flattening theorem of Huang-Krantz [HK].

We next state our main results, in which we will use some terminology to be defined in the next section:

Theorem 1.1: *Let M be a formal Bishop surface with an elliptic complex tangent at 0, whose Bishop invariant λ is 0 and whose Moser invariant s is a finite integer greater than two . Then there exists a formal transformation,*

$$(z', w') = F(z, w) = (\widetilde{f}(z, w), \widetilde{g}(z, w)), \quad F(0, 0) = (0, 0)$$

such that in the (z', w') coordinates, $M' = F(M)$ is represented near the origin by a formal equation of the following normal form:

$$w' = z'\bar{z'} + z'^s + \bar{z'}^s + \varphi(z') + \overline{\varphi(z')}$$

where

$$\varphi(z') = \sum_{k=1}^{\infty}\sum_{j=2}^{s-1} a_{ks+j} z'^{ks+j}.$$

Such a formal transform is unique up to a composition from the left with a rotation of the form:

$$z'' = e^{i\theta} z',\ w'' = w', \quad \text{where } \theta \text{ is a constant with } e^{is\theta} = 1.$$

Theorem 1.2: *Let M and M' be real analytic Bishop surfaces near 0 with the Bishop invariant vanishing and the Moser invariant finite. Suppose that $F : (M, 0) \longrightarrow (M', 0)$ is a formal equivalence map. Then F is biholomorphic near 0.*

Define $\mathcal{Z}_s$ for the group of transformations consisting of maps of the form $\{\psi_\theta : (z, w) \mapsto (e^{i\theta} z, w),\ e^{is\theta} = 1\}$.

We next give several immediate consequences of Theorems 1.1 and 1.2:

Corollary 1.3: *(a): Suppose M_{nor} is a formal Bishop surface near the origin defined by*

$$w = z\bar{z} + z^s + \bar{z}^s + 2Re\{\sum_{k=1}^{\infty}\sum_{j=2}^{s-1} a_{ks+j} z^{ks+j}\}.$$

Then the group of the origin preserving formal self-transformations of M_{nor}, denoted by $aut_0(M_{nor})$, is a subgroup of $\mathcal{Z}_s$. Moreover, $\psi_\theta \in aut_0(M_{nor})$ if and only if

$$a_{ks+j} = 0 \ \text{ for any } k \text{ and } j \text{ with } k \geq 1,\ 2 \leq j \leq s-1, \quad e^{\sqrt{-1}j\theta} \neq 1.$$

(b): $aut_0(M_s) = \mathcal{Z}_s$, where M_s is defined by $w = z\bar{z} + z^s + \bar{z}^s$.

(c): Any subgroup of $\mathcal{Z}_s$ can be realized as the formal automorphism group of a certain M_{nor}.

(d): Let M be a formal Bishop surface with a vanishing Bishop invariant and $s < \infty$ at 0. Then $aut_0(M)$ is isomorphic to a subgroup of $\mathcal{Z}_s$.

(e): Let M be a real analytic Bishop surface with a vanishing Bishop invariant and the Moser invariant $s < \infty$ at 0. Suppose that $aut_0(M) = \mathcal{Z}_s$. Then $(M, 0)$ is biholomorphic to $(M_s, 0)$, where M_s, as before, is defined by $w = z\bar{z} + z^s + \bar{z}^s$.

(f): Let M be a real analytic elliptic Bishop surface with $\lambda = 0$ and s a prime number at 0. Then $aut_0(M)$ is a trivial group unless $(M, 0)$ is biholomorphic to $(M_s, 0)$.

Corollary 1.4: *Let M_1 and M_2 be real analytic Bishop surfaces with $\lambda = 0$ and $s \neq \infty$ at 0. Suppose that M_1 has a formal normal form:*

$$w' = z'\bar{z}' + z'^s + \bar{z}'^s + 2Re\{\sum_{k=1}^{\infty}\sum_{j=2}^{s-1} a_{ks+j} z'^{ks+j}\};$$

and suppose that M_2 has a formal normal form:

$$w' = z'\bar{z}' + z'^s + \bar{z}'^s + 2Re\{\sum_{k=1}^{\infty}\sum_{j=2}^{s-1} b_{ks+j}{z'}^{ks+j}\}.$$

Then $(M_1, 0)$ is biholomorphic to $(M_2, 0)$ if and only if there is a constant θ, with $e^{s\theta\sqrt{-1}} = 1$, such that $a_{ks+j} = e^{\theta j\sqrt{-1}} b_{ks+j}$ for any $k \geq 1$ and $j = 2, \cdots, s-1$.

Theorem 1.5: *A generic real analytic Bishop surface with a vanishing Bishop invariant and $s \neq \infty$ is not holomorphically equivalent to an algebraic surface in $\mathbb{C}^2$.*

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2 Uniqueness of formal maps between approximately normalized surfaces

In what follows, we use (z, w) or (z', w') for the coordinates for $\mathbb{C}^2$. Let $A(z, \overline{z})$ be a formal power series in $(z, \overline{z})$ without constant term. We say that the order of $A(z, \bar{z})$ is k if $A(z, \overline{z}) = \sum_{j+l=k} A_{j\bar{l}} z^j \overline{z}^l + o(|z|^k)$ with at least one of the $A_{j\bar{l}} \in \mathbb{C}$ $(j + l = k)$ not equal to 0. In this case, we write $\mathrm{Ord}(A(z, \overline{z})) = k$. We say $\mathrm{Ord}(A(z, \overline{z})) \geq k$ if $A(z, \overline{z}) = O(|z|^k)$.

Consider a formal real surface M in $\mathbb{C}^2$ near the origin. Suppose that 0 is a point of complex tangent for M. Then, after a linear change of variables, we can assume that $T_0^{(1,0)} M = \{w = 0\}$. If there is no change of coordinates such that M is defined by an equation of the form $w = O(|z|^3)$, we then say 0 is a point of M with a non-degenerate complex tangent. In this case, Bishop showed that there is a change of coordinates in which M is defined by ([Bis] [Hu1])

$$w = z\overline{z} + \lambda(z^2 + \overline{z}^2) + O(|z|^3). \tag{2.1}$$

Here $\lambda \in [0, \infty]$ and when $\lambda = \infty$, the equation takes the form: $w = z^2 + \overline{z}^2 + O(|z|^3)$. λ is the first absolute invariant of M at 0, called the Bishop invariant. Bishop invariant is a quadratic invariant, resembling to the Levi eigenvalue in the hypersurface case. When $\lambda \in [0, 1/2)$, we say that M has an elliptic complex tangent at 0. In this paper, we are only interested in the case of an elliptic complex tangent. We need only to study the case of $\lambda = 0$; for, in the case

with $\lambda \in (0, 1/2)$, the surface has been well understood by the work of Moser-Webster [MW]. When $\lambda = 0$, Moser-Webster and Moser showed in [MW] [Mos] that there is an integer $s \geq 3$ or $s = \infty$ such that M is defined by

$$w = z\overline{z} + z^s + \overline{z}^s + E(z, \overline{z}), \tag{2.2}$$

where E is a formal power series in $(z, \overline{z})$ with $\mathrm{Ord}(E) \geq s+1$. When $s = \infty$, we understand the defining equation as $w = z\overline{z}$, namely, M is formally equivalent to the quadric $M_\infty = \{w = z\overline{z}\}$. s is the next absolute invariant for M, called the Moser invariant. The case for $s = \infty$ is also well-understood through the work of Moser [Mos]. Hence, in all that follows, our M will have $\lambda = 0$ and a fixed $s < \infty$.

A formal map $z' = F(z, w)$, $w' = G(z, w)$ without constant terms is called an invertible formal transformation (or simply, a formal transformation) if $\frac{\partial(F,G)}{\partial(z,w)}(0, 0)$ is invertible. When a formal map has no constant term, we also say that it preserves the origin.

Lemma 2.1: *Let M be defined as in (2.2). Suppose that $z' = F(z, w), w' = G(z, w)$ is a formal transformation preserving the origin and sending M into M', which is defined by $w' = z'\overline{z}' + E^*(z', \overline{z'})$. Then*
(i): $F = az + bw + O(|(z, w)|^2)$, $G = cw + O(|w|^2 + |zw| + |z|^3)$ where $c = |a|^2, a \neq 0$.
(ii): Suppose that M and M' are further defined by $w = E(z, \overline{z}) = z\overline{z} + z^s + \overline{z}^s + o(|z|^s)$ and $w' = E^*(z', \bar{z'}) = z'\overline{z'} + z'^s + \overline{z'}^s + o(|z'|^s)$, respectively. Here $s \geq 3$. Then

$$F = (e^{i\theta}z + O(|z|^2 + |w|), w + O(|w|^2 + |zw| + |z|^3)), \quad \text{where } \theta \text{ is a constant with } \ e^{is\theta} = 1.$$

(iii): In (i), when $\overline{E(z, \overline{z})} = E(z, \overline{z})$ and $\overline{E^*(z', \overline{z}')} = E^*(z', \overline{z}')$, we have $G(z, w) = G(0, w)$.

Proof of Lemma 2.1: (i) is the content of Lemma 3.2 of [Hu1]. To prove (ii), we write $F = (az + f, cw + g)$, where by (i), we can assume that

$$f(z, w) = O(|z|^2 + |w|) \quad , \quad g(z, w) = O(|w|^2 + |zw| + |z|^3).$$

Notice that

$$f(0, E(0, \bar{z})) = O(\bar{z}^s) \quad , \quad \overline{f}(\bar{z}, \bar{E}(\bar{z}, 0)) = O(\bar{z}^2) \quad , \quad g(0, E(0, \bar{z})) = o(\bar{z}^s).$$

Applying the defining equation of M' , we have, on M, the following:

$$\begin{aligned} cw + g(z, w) \quad &= \quad |a|^2|z|^2 + \bar{a}\bar{z}f(z, w) + az\overline{f}(\bar{z}, \bar{w}) + f(z, w)\overline{f}(\bar{z}, \bar{w}) \\ &\quad + (az + f(z, w))^s + \big(\bar{a}\bar{z} + \overline{f}(\bar{z}, \bar{w})\big)^s + o(|z|^s). \end{aligned}$$

Regarding z and $\bar{z}$ as independent variables in the above equation and then letting $z = 0, w = E(0, \bar{z}), \bar{w} = \bar{E}(\bar{z}, 0)$, we obtain

$$c\bar{z}^s + o(\bar{z}^s) = (\bar{a}\bar{z})^s + o(\bar{z}^s).$$

Hence, it follows that $c = \bar{a}^s$. Together with $c = |a|^2$ and $s \geq 3$, we get

$$c = 1 \ , \ a = e^{i\theta} \ , \text{ where } \theta \text{ is a constant.}$$

Now we turn to the proof of (iii). Notice that

$$G(z,w) = |F(z,w)|^2 + E^*(F(z,w), \overline{F(z,w)}) \ for \ (z,w) \in M.$$

Since E^* is now assumed to be formally real valued, we have

$$G(z,w) = \overline{G(z,w)} \quad \text{on } M.$$

Write

$$G(z,w) = \sum_{\alpha,\beta}^{\infty} a_{\alpha\beta} z^\alpha w^\beta.$$

We will prove inductively that $a_{\alpha\beta} = \overline{a_{\alpha\beta}}$ for $\alpha = 0$ and $a_{\alpha\beta} = 0$ otherwise. First, for each $m >> 1$, write $E = E_{(m)}(z, \overline{z}) + E_m$ with $E_{(m)}(z, \overline{z})$ a polynomial of degree at most $m-1$ and $E_m = O(|z|^m)$. Then for any $m >> 1$, there are integers $N_1(m) >> m$ and $N_2(m) >> m$ such that

$$\sum_{\alpha,\beta=0}^{N_2(m)} a_{\alpha\beta} z^\alpha w^\beta = \sum_{\alpha,\beta=0}^{N_2(m)} \overline{a_{\alpha\beta} z^\alpha} w^\beta + o(|z|^m) \quad , \quad w = z\overline{z} + E_{(N_1(m))}(z, \overline{z}). \tag{2.3}$$

Next, suppose that $N_0 = \alpha_0 + 2\beta_0$ is the smallest number such that $a_{\alpha\beta}$ is real-valued for $\alpha = 0$, and zero otherwise whenever $\alpha + 2\beta < N_0$. (If such an N_0 does not exist, then Lemma 2.1 (iii) holds automatically). Choose $m >> N_0$. For $0 < r << 1$, define $\sigma_{N_1}(\xi, r)$ to be the biholomorphic map from the unit disk in $\mathbb{C}$ to the smoothly bounded simply connected domain: $\{\xi \in \mathbb{C} : \ |\xi|^2 + r^{-2} E_{(N_1)}(r\xi, r\overline{\xi}) < 1\}$ with $\sigma_{N_1}(\xi, r) = \xi(1 + O(r))$. Since the disk $(r\sigma_{N_1}(\xi, r), r^2)$ is attached to M_{N_1} defined by $w = z\overline{z} + E_{(N_1)}(z, \overline{z})$, it follows that

$$\sum_{\alpha+2\beta=N_0} a_{\alpha\beta} r^{N_0} \xi^\alpha = \sum_{\alpha+2\beta=N_0} \overline{a_{\alpha\beta} \xi^\alpha} r^{N_0} + o(r^{N_0}), \quad |\xi| = 1. \tag{2.4}$$

Letting $r \to 0$, we get

$$\sum_{\alpha+2\beta=N_0} a_{\alpha\beta} \xi^\alpha = \sum_{\alpha+2\beta=N_0} \overline{a_{\alpha\beta} \xi^\alpha}, \quad |\xi| = 1, \tag{2.5}$$

from which we see that when $\alpha + 2\beta = N_0$, $a_{\alpha\beta}$ is real for $\alpha = 0$, and zero otherwise. This contradicts the choice of N_0 and thus completes the proof of Lemma 2.1 (iii). ∎

The main purpose of this section is to prove the following uniqueness result for mappings between approximately normalized surfaces:

Theorem 2.2: *Suppose that the formal power series*

$$\begin{cases} z' = z + f(z,w), & f(z,w) = O(|w| + |z|^2) \\ w' = w + g(w), & g(w) = O(|w|^2) \end{cases} \tag{2.6}$$

transforms the formal Bishop surface M defined by

$$w = z\overline{z} + 2Re\left(z^s + \sum_{k=1}^{n}\sum_{j=2}^{s-1} a_{ks+j} z^{ks+j}\right) + E_1(z,\overline{z})$$

to the formal Bishop surface defined by

$$w' = z'\bar{z'} + 2Re\left(z'^s + \sum_{k=1}^{n}\sum_{j=2}^{s-1} b_{ks+j} z'^{ks+j}\right) + E_2(z',\overline{z'})$$

where $n \geq 1$, a_{ks+j}, b_{ks+j} are complex numbers, and $E_1(z,\overline{z})$, $E_2(z,\overline{z}) = o(|z|^{ns+s-1})$. Then $f(tz,t^2w) = O(t^{2n+1})$, $g(t^2w) = O(t^{2n+2})$, as $t \in \mathbf{R} \to 0$, and $a_{ks+j} = b_{ks+j}$ for all $k \leq n$ and $j = 2,\cdots,s-1$.

One of the crucial ideas for the proof of Theorem 2.2 is to set the weight of $\overline{z}$ differently from that of z. More precisely, we set the weight of z to be 1 and that of $\bar{z}$ to be $s-1$. For a formal power series $A(z,\overline{z})$ with no constant term, we say that $wt(A(z,\overline{z})) = k$, or $wt(A(z,\overline{z})) \geq k$, if $A(tz,t^{s-1}\overline{z}) = t^kA(z,\overline{z})$, or , $A(tz,t^{s-1}\overline{z}) = O(t^k)$, respectively, as $t \in \mathbb{R} \to 0$. In all that follows, we use Θ_l^j to denote a formal power series in z and $\bar{z}$ of order at least j and weight at least l. (Namely, $\Theta_l^j(tz,t\overline{z}) = O(t^j)$ and $\Theta_l^j(tz,t^{s-1}\overline{z}) = O(t^l)$ as $t \to 0$). We use $\mathbb{P}_l^j$ to denote a homogeneous polynomial in z and $\bar{z}$ with the exact order j and weight at least l. *We emphasize that Θ_l^j and $\mathbb{P}_l^j$ may be different in different contexts.*

In what follows, we also define the normal weight of z,w to be $1,2$, respectively. For a formal power series $h(z,w,\overline{z},\overline{w})$, we use $wt_{nor}(h) \geq k$ to denote the vanishing property: $h(tz,t^2w,t\overline{z},t^2\overline{w}) = O(t^k)$ as $t \to 0$. Let $h(z,w)$ be a formal power series in (z,w) without constant term. Then we have the formal expansion:

$$h(z,w) = \sum_{l=1}^{\infty} h_{nor}^{(l)}(z,w)$$

where

$$h_{nor}^{(l)}(tz,t^2w) = t^l h_{nor}^{(l)}(z,w)$$

is a polynomial in (z,w). Notice that $h_{nor}^{(l)}(z,w)$ is homogeneous of degree l in the standard weighting system which assigns the weight of z and w to be 1 and 2, respectively. In what follows, we write

$$h_l(z,w) = \sum_{j=l}^{\infty} h_{nor}^{(j)}(z,w) \quad \text{and} \quad h_{(l)} = \sum_{j=1}^{l-1} h_{nor}^{(j)}(z,w). \tag{2.7}$$

Proof of Theorem 2.2: We need to prove that any solution (f,g) of the following equation has the property that $wt_{nor}(f(z,w)) \geq 2n+1$, $wt_{nor}(g(w)) \geq 2n+2$ under the normalization

conditions as in the theorem:

$$
\begin{aligned}
w+g(w) &= (z+f(z,w))(\bar{z}+\overline{f(z,w)})+2Re\{(z+f(z,w))^s \\
&\quad +\sum_{k=1}^{n}\sum_{j=2}^{s-1} b_{ks+j}(z+f(z,w))^{ks+j}\}+E_2(f(z,w),\overline{f(z,w)})
\end{aligned}
\tag{2.8}
$$

where $w = z\bar{z}+z^s+\bar{z}^s+E(z,\bar{z})$ with

$$
E = 2Re\left(\sum_{k=1}^{n}\sum_{j=2}^{s-1} a_{ks+j}z^{ks+j}\right)+E_1(z,\overline{z}).
$$

With an immediate simplification, (2.8) takes the form:

$$
\begin{aligned}
g(w) &= \bar{z}f(z,w)+z\overline{f(z,w)}+|f(z,w)|^2+2Re\big\{(z+f(z,w))^s-z^s \\
&\quad +\sum_{k=1}^{n}\sum_{j=2}^{s-1}\big(b_{ks+j}(z+f(z,w))^{ks+j}-a_{ks+j}z^{ks+j}\big)\big\}+o(|z|^{ns+s-1})
\end{aligned}
\tag{2.9}
$$

In the proof of Theorem 2.2, we set the following convention. For any positive integer N, we define a_N and b_N to be as in Theorem 2.2 if $N = ks+j$ with $k \le n,\ 2 \le j \le s-1$, and to be 0 otherwise. For the rest of this section, we will define a positive integer N_0 as follows:

Suppose that there is a pair of integers (j_0, k_0) such that $s < k_0 s + j_0 (\le ns+s-1)$ is the smallest number satisfying $a_{k_0 s+j_0} \ne b_{k_0 s+j_0}$. We then define $N_0 = k_0 s + j_0$. Otherwise, we define $N_0 = sn+s$.

The proof of Theorem 2.2 is carried out in two steps, according to the vanishing order of f being even or odd.

Step I of the proof of Theorem 2.2: In this step, we assume that either

$$
\mathrm{Ord}\,(f(z,w(z,\bar{z}))=2t
$$

is an even number or $f \equiv 0$, where $w(z,\bar{z}) = z\bar{z}+z^s+\bar{z}^s+E(z,\bar{z})$. Write $g(w) = c_l w^l + o(w^l)$.

Denote by $\widehat{N_0} = \min\{N_0,\ \mathrm{Ord}(f), sn+s-1\}$. (If $f \equiv 0$, we define $\mathrm{Ord}(f) = \infty$.) Then (2.9) gives the following:

$$
c_l z^l \bar{z}^l + O(|z|^{2l+1}) = 2Re[(b_{N_0}-a_{N_0})z^{N_0}]+O(|z|^{\widehat{N_0}+1}).
\tag{2.10}
$$

From this, we can easily conclude the following:

(2.I). Suppose that $2t \ge N_0$ and $c_l \ne 0$. Then $2l > \min\{N_0, sn+s-1\}$ and $b_{N_0} = a_{N_0}$. By our choice of N_0, N_0 must be $ns+s$. Hence, the theorem in this case readily follows.

(2.II). When $2t < N_0$, then $2l \ge \min\{2t+2, sn+s\}$ under the assumption that $c_l \ne 0$. Thus $l > t \ge 1$ (if $c_l \ne 0$).

Suppose that $N_0 = 2t+1$ in Case (2.II). Assuming that $N_0 < ns+s$ and collecting terms with degree $2t+1$ in (2.9), we obtain

$$\bar{z}f_{nor}^{(2t)}(z,z\bar{z}) + z\overline{f_{nor}^{(2t)}(z,z\bar{z})} + 2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right) = 0 \tag{2.11}$$

This clearly forces that $a_{N_0} = b_{N_0}$. Thus, we must have $N_0 = ns+s$ and Theorem 2.2 also follows easily in this setting. *Hence, we will assume, in what follows:*

(2.III). $ns+s > N_0 \geq 2t+2,\ l > t \geq 1.$

Collecting terms with (the ordinary) degree 2t+1 in (2.9), we get:

$$\bar{z}f_{nor}^{(2t)}(z,z\bar{z}) + z\overline{f_{nor}^{(2t)}(z,z\bar{z})} = 0 \tag{2.12}$$

Writing $f_{nor}^{(2t)}(z,w) = \sum_{k+2l=2t} a_{kl}z^k w^l$ and substituting it back to (2.12), we then get:

$$f_{nor}^{(2t)}(z,w) = aw^t - \bar{a}z^2w^{t-1}$$

for $a \neq 0$. Hence

$$f(z,w) = f_{nor}^{(2t)}(z,w) + f_{2t+1}(z,w) = aw^t - \bar{a}z^2w^{t-1} + f_{2t+1}(z,w) \tag{2.13}$$

Next, a simple computation shows that $wt(w) \geq s$, $\mathrm{Ord}(w(z,\overline{z})) \geq 2$, $wt(f_{nor}^{(2t)}) \geq st+2-s$, $wt(\overline{f_{nor}^{(2t)}}) \geq st$, $g = g_{2t+2}$, $f = f_{nor}^{(2t)} + f_{2t+1}(z,w)$. Also if $l_1+l_2 \geq s$ with $l_2 > 1$, or $l_1+l_2 > s$ with $l_2 \geq 1$, then $wt(z^{l_1}f_{nor}^{(2t)l_2}) = l_1 + l_2(ts+2-s) \geq ts+2$. Moreover, $wt(z^{l_1}f_{nor}^{(2t)l_2}f_{2t+1}^{l_3}) \geq s$ if $l_1+l_2+l_3 \geq s-1$, $l_2^2+l_3^2 \neq 0$.

We can verify the following

$$|f(z,w)|^2 = 2Re(\overline{f_{nor}^{(2t)}}f_{2t+1}) + \Theta_{st+2}^{2t+2} + \Theta_{st+2}^2 f_{2t+1}.$$

Substituting (2.13) into (2.9), we get:

$$\begin{aligned} g_{2t+2}(w) &= 2Re\{(\overline{z}+sz^{s-1})f\} + |f(z,w)|^2 + 2Re\{\textstyle\sum_{l=2}^{s}\mathbb{P}_{s-l}^{s-l}f^l\} \\ &\quad +2Re(\sum_{\tau=ks+j<N_0}\textstyle\sum_{l=0}^{\tau-1}\mathbb{P}_l^l f^{\tau-l}) + 2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right) + \Theta_{N_0+1}^{N_0+1} \\ &= 2Re\{(\bar{z}+sz^{s-1})f_{nor}^{(2t)} + (\bar{z}+sz^{s-1}+\overline{f_{nor}^{(2t)}})f_{2t+1}(z,w)\} \\ &\quad +2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right) + \Theta_s^2 f_{2t+1}(z,w) + \Theta_s^2\overline{f_{2t+1}(z,w)} + \Theta_{N_s}^{2t+2} \end{aligned} \tag{2.14}$$

Here N_0 is defined as before and $N_s := \min\{ts+2, N_0+1\}$.

Notice that

$$\begin{aligned} &\bar{z}f_{nor}^{(2t)} + z\overline{f_{nor}^{(2t)}} + 2Re\{sz^{s-1}f_{nor}^{(2t)}\} \\ &= 2Re\{\bar{z}(aw^t - \bar{a}z^2w^{t-1}) + sz^{s-1}(aw^t - \bar{a}z^2w^{t-1})\} \\ &= -\bar{a}z^2\bar{z}w^{t-1} + z\bar{a}w^t - sz^{s-1}\bar{a}z^2w^{t-1} + \Theta_{ts+2}^{2t+2} \\ &= (1-s)\bar{a}z^{s+1}w^{t-1} + \Theta_{ts+2}^{2t+2} \end{aligned} \tag{2.15}$$

Hence, we obtain

$$
\begin{aligned}
g_{2t+2}(w) \; = \; & (1-s)\bar{a}z^{s+1}(z\bar{z}+z^s)^{t-1} + (\bar{z}+sz^{s-1}+\Theta_s^2)f_{2t+1}(z,w) \\
& +2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right) + (z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{2t+1}(z,w)} \\
& +2Re\{\overline{f_{nor}^{(2t)}}f_{2t+1}(z,w)\} + \Theta_{N_s}^{2t+2}
\end{aligned} \tag{2.16}
$$

If $t=1$, collecting terms of degree $s+1$ in (2.16) and noticing that $N_0 > s+1$ by the given condition, we get

$$
\begin{aligned}
\sum_{2j}\delta_{2j}^{s+1}g_{nor}^{(2j)}(z\overline{z}) \; = \; & (1-s)\bar{a}z^{s+1} + \bar{z}f_{nor}^{(s)}(z,z\overline{z}) + z\overline{f_{nor}^{(s)}(z,z\overline{z})} \\
& +az^2\overline{f_{nor}^{(s-1)}(z,z\overline{z})} + \overline{az^2}f_{nor}^{(s-1)}(z,z\overline{z}) + \mathbb{P}_{s+2}^{s+1}.
\end{aligned} \tag{2.17}
$$

Here δ_{2j}^{s+1} takes value 1, when $2j=s+1$, and 0 otherwise.

Since $s+2 \geq s+1$, $\mathbb{P}_{s+2}^{s+1} = \bar{z}A$ with A a polynomial. Thus it follows easily that $(1-s)\bar{a}z^{s+1}$ divides $\bar{z}$. This is a contradiction and thus $t>1$. In particular, (2.16) can be written as

$$
\begin{aligned}
g_{2t+2}(w) \; = \; & (1-s)\bar{a}z^{s+1}(z\bar{z}+z^s)^{t-1} + (\bar{z}+sz^{s-1}+\Theta_s^2)f_{2t+1}(z,w) \\
& +2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right) + (z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{2t+1}(z,w)} + \Theta_{N_s}^{2t+2}
\end{aligned} \tag{2.18}
$$

We next prove the following:

Lemma 2.3: Assume that $2t+j(s-2)+2 \leq m \leq 2t+(j+1)(s-2)+1$ with $0 \leq j \leq t-1$ and $m \leq N_0$. Then

$$
\begin{aligned}
g_m(w) \; = \; & \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1} + (\bar{z}+sz^{s-1}+\Theta_s^2)f_{m-1}(z,w) \\
& +(z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{m-1}(z,w)} + 2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right) + \Theta_{N_s}^m
\end{aligned} \tag{2.19}
$$

Proof of Lemma 2.3: The argument presented above gives the proof of the lemma with $m = 2t+2$. We complete the proof of the lemma in three steps.

Step I of the proof of Lemma 2.3: This step is not needed when $s=3$. Denote $m_0 = 2t+j(s-2)+2$, where j is an integer with $0 \leq j \leq t-1$. Suppose that $m_0 \leq N_0$. We also assume that there is an integer m such that $m \geq m_0$, $m+1 \leq 2t+(j+1)(s-2)+1$ (such an m certainly does not exist if $s=3$), $m+1 \leq N_0$ and moreover the formula (2.19) holds for this m. Collecting terms of degree m in (2.19), we get

$$
g^{(m)}(z\bar{z}) = \bar{z}f_{nor}^{(m-1)}(z,z\bar{z}) + z\overline{f_{nor}^{(m-1)}(z,z\bar{z})} + \hat{\mathbb{P}}_{N_s}^m \tag{2.20}
$$

Notice that $\hat{\mathbb{P}}_{N_s}^m (= \mathbb{P}_{N_s}^m)$ must be real valued, and notice that $g^{(m)}(z\overline{z})$ is also of weight at least N_s. We can write

$$
g^{(m)}(z\bar{z}) - \mathbb{P}_{ts+2}^m = \sum_{\substack{\alpha+\beta=m \\ \alpha+\beta(s-1)\geq N_s}} a_{\alpha\bar{\beta}}z^{\alpha}\bar{z}^{\beta} \tag{2.21}
$$

Write

$$f_{nor}^{(m-1)}(z,z\overline{z}) = \sum_{\widetilde{\alpha}+2\widetilde{\beta}=m-1} b_{\widetilde{\alpha}\widetilde{\beta}} z^{\widetilde{\alpha}}(z\bar{z})^{\widetilde{\beta}} = \sum_{\widetilde{\alpha}+2\widetilde{\beta}=m-1} b_{\widetilde{\alpha}\widetilde{\beta}} z^{\widetilde{\alpha}+\widetilde{\beta}}\overline{z}^{\widetilde{\beta}}. \tag{2.22}$$

Then

$$\sum_{\widetilde{\alpha}+2\widetilde{\beta}=m-1} b_{\widetilde{\alpha}\widetilde{\beta}} z^{\widetilde{\alpha}+\widetilde{\beta}}\overline{z}^{\widetilde{\beta}+1} + \sum_{\widetilde{\alpha}+2\widetilde{\beta}=m-1} \overline{b_{\widetilde{\alpha}\widetilde{\beta}}}\overline{z}^{\widetilde{\alpha}+\widetilde{\beta}} z^{\widetilde{\beta}+1} = \sum_{\substack{\alpha+\beta=m \\ \alpha+\beta(s-1)\geq N_s}} a_{\alpha\bar{\beta}} z^{\alpha}\overline{z}^{\beta} \tag{2.23}$$

We see that if m is even, then $2b_{\widetilde{\alpha}\widetilde{\beta}} = a_{\alpha\bar{\beta}} + ic$ when $\alpha = \beta = \frac{m}{2},\ \widetilde{\alpha} = 1,\ \widetilde{\beta} = \frac{m}{2} - 1,\ c \in \mathbb{R}$. The other relations are as follows:

$$b_{\widetilde{\alpha}\widetilde{\beta}} = b_{\alpha\bar{\beta}}, \ \text{ if } \widetilde{\alpha} + \widetilde{\beta} = \alpha,\ \widetilde{\alpha} + 2\widetilde{\beta} = m-1,\ \widetilde{\beta}+1 = \beta, \widetilde{\alpha} > 1,\ \alpha + (s-1)\beta \geq N_s. \tag{2.24}$$

From this, one can easily see that

$$wt(f_{nor}^{(m-1)}(z,\overline{z})) \geq \min\{\widetilde{\alpha} + \widetilde{\beta} + (s-1)\widetilde{\beta}\} = \min\{\alpha + (s-1)\beta - s + 1\} \geq N_s - s + 1. \tag{2.25}$$

Substituting (2.22) into (2.19), we get

$$\begin{aligned} g_{m+1}(w) \ &= \ (1-s)^{j+1}\bar{a}z^{(j+1)s+1}(z\bar{z} + z^s)^{t-j-1} + (\bar{z} + sz^{s-1} + \Theta_s^2) f_m(z,w) \\ &\quad +(z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_m(z,w)} + \Theta_{N_s}^{m+1} + (sz^{s-1} + \Theta_s^2) f_{nor}^{(m-1)} \\ &\quad +2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + (s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{nor}^{(m-1)}} \end{aligned} \tag{2.26}$$

By (2.25), we get

$$(sz^{s-1} + \Theta_s^2) f_{nor}^{(m-1)} + (s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{nor}^{(m-1)}} = \mathbb{P}_{N_s}^{m+1}.$$

Hence

$$\begin{aligned} g_{m+1}(w) \ &= \ (1-s)^{(j+1)}\bar{a}z^{(j+1)s+1}(z\bar{z} + z^s)^{t-j-1} + (\bar{z} + sz^{s-1} + \Theta_s^2) f_m(z,w) \\ &\quad +(z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_m(z,w)} + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + \Theta_{N_s}^{m+1}. \end{aligned} \tag{2.27}$$

By induction, we showed that if the lemma holds for m_0 defined above, then it holds for any m with $m_0 \leq m \leq 2t + (j+1)(s-2) + 1$ and $m \leq N_0$.

Step II of the proof of Lemma 2.3: In this step, suppose that we know that the lemma holds for $m \in [2t + j(s-2) + 2, 2t + (j+1)(s-2) + 1]$ with $m \leq N_0$, where j is a certain non-negative integer bounded by $t-2$. We then proceed to prove that the lemma holds also for $m \in [2t + (j+1)(s-2) + 2, 2t + (j+2)(s-2) + 1]$, whenever $m \leq N_0$.

Suppose that $2t + (j+1)(s-2) + 1 < N_0$. By the assumption, we have

$$\begin{aligned} g_{2t+(j+1)(s-2)+1}(w) \ &= \ \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z} + z^s)^{t-j-1} \\ &\quad +(\bar{z} + sz^{s-1} + \Theta_s^2) f_{2t+(j+1)(s-2)}(z,w) \\ &\quad +(z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{2t+(j+1)(s-2)}(z,w)} \\ &\quad +2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + \Theta_{N_s}^{2t+(j+1)(s-2)+1}. \end{aligned} \tag{2.28}$$

Collecting terms of degree $2t+(j+1)(s-2)+1$ in (2.28), we get

$$\begin{array}{rcl} g^{(2t+(j+1)(s-2)+1)}(z\bar{z}) & = & \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1}+\hat{\mathbb{P}}_{N_s}^{2t+(j+1)(s-2)+1} \\ & & +\bar{z}f_{nor}^{(2t+(j+1)(s-2))}(z,z\bar{z})+\overline{zf_{nor}^{(2t+(j+1)(s-2))}(z,z\bar{z})}. \end{array} \tag{2.29}$$

Here we denote by $\hat{\mathbb{P}}_{N_s}^{2t+(j+1)(s-2)+1}$ a certain homogeneous polynomial of degree $2t+(j+1)(s-2)+1$ with weight at least N_s.

Now, we solve (2.29) as follows. Write $\Lambda = 2t+(j+1)(s-2)$. Notice that

$$I := -\hat{\mathbb{P}}_{N_s}^{\Lambda+1}+a(1-s)^{j+1}\bar{z}^{(j+1)s+1}(z\bar{z})^{t-j-1}+g^{(\Lambda+1)}(z\bar{z})$$

is real valued and $I=\mathbb{P}_{N_s}^{\Lambda+1}$. Then (2.29) can be rewritten as

$$\begin{array}{rcl} I & = & \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1}+a(1-s)^{j+1}\bar{z}^{(j+1)s+1}(z\bar{z})^{t-j-1} \\ & & +\bar{z}f_{nor}^{(2t+(j+1)(s-2))}(z,z\bar{z})+\overline{zf_{nor}^{(2t+(j+1)(s-2))}(z,z\bar{z})}. \end{array} \tag{2.30}$$

Write

$$I=\sum_{\substack{l+k=\Lambda+1\\ l+(s-1)k\geq N_s}} a_{l\bar{k}}z^l\bar{z}^k.$$

Since $a_{l\bar{k}}=\overline{a_{k\bar{l}}}$, we also require that $k+(s-1)l\geq N_s$.

We next can get the following general solution of (2.30):

$$\begin{array}{l} f_{nor}^{(2t+(j+1)(s-2))}(z,w)=f_1^{(\Lambda)}+f_2^{(\Lambda)} \quad \text{with} \\ f_1^{(\Lambda)}=-\bar{a}(1-s)^{j+1}z^{(j+1)s+2}w^{t-j-2} \\ f_2^{(\Lambda)}=\sum_{\widetilde{l}+2\widetilde{k}=\Lambda}h_{\widetilde{l}\widetilde{k}}z^{\widetilde{l}}w^{\widetilde{k}} \end{array} \tag{2.31}$$

where $h'_{\widetilde{l}\widetilde{k}}s$ are determined by the following:

$$\sum h_{\widetilde{l}\widetilde{k}}z^{\widetilde{l}+\widetilde{k}}\bar{z}^{\widetilde{k}+1}+\sum\overline{h_{\widetilde{l}\widetilde{k}}}z^{\widetilde{l}+\widetilde{k}+1}\bar{z}^{\widetilde{k}}=\sum_{l,k}a_{lk}z^l\bar{z}^k. \tag{2.32}$$

Hence, we see that if $h_{\widetilde{l}\widetilde{k}}\neq 0$, then either $\widetilde{l}=1,\ 2\widetilde{k}=\Lambda$ (in case Λ is even) or $\widetilde{l}+\widetilde{k}=l,\ \widetilde{k}+1=k$. Here $l,\ k$ satisfy the properties described above. Based on such an analysis and as argued before, we can conclude the following:

$$(sz^{s-1}+\Theta_s^2)f_2^{(\Lambda)}(z,z\bar{z})+(s\bar{z}^{s-1}+\Theta_s^2)\overline{f_2^{(\Lambda)}(z,z\bar{z})}=\Theta_{N_s}^{\Lambda+2}. \tag{2.33}$$

Hence, from (2.28)-(2.33), we get

$$\begin{array}{rcl} g_{\Lambda+2}(w)+g_{nor}^{(\Lambda+1)}(w) & = & (\bar{z}+sz^{s-1}+\Theta_s^2)f_{\Lambda+1}(z,w)+(z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{\Lambda+1}(z,w)} \\ & & +\Theta_{N_s}^{\Lambda+2}+\hat{\mathbb{P}}_{N_s}^{\Lambda+1}+\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1} \\ & & +(\bar{z}+sz^{s-1}+\Theta_s^2)f_{nor}^{(\Lambda)}(z,w)+(z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{nor}^{(\Lambda)}(z,w)} \\ & & +2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right). \end{array} \tag{2.34}$$

Notice that

$$g_{nor}^{(\Lambda+1)}(z\bar{z}) = \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} + \bar{z}f_{nor}^{(\Lambda)}(z, z\bar{z}) + z\overline{f_{nor}^{(\Lambda)}(z, z\bar{z})} + \hat{\mathbb{P}}_{N_s}^{\Lambda+1},$$
$$g_{nor}^{(\Lambda+1)}(w) - g_{nor}^{(\Lambda+1)}(z\bar{z}) \in \Theta_{N_s}^{\Lambda+2}.$$

We get

$$\begin{aligned} g_{\Lambda+2}(w) = \ & (\bar{z} + sz^{s-1} + \Theta_s^2)f_{\Lambda+1}(z,w) + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) \\ & +(z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{\Lambda+1}(z,w)} + \Theta_{N_s}^{\Lambda+2} + J, \end{aligned} \tag{2.35}$$

where

$$\begin{aligned} J = \ & (\bar{z} + sz^{s-1} + \Theta_s^2)f_{nor}^{(\Lambda)}(z,w) + (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{nor}^{(\Lambda)}(z,w)} \\ & +\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z} + z^s)^{t-j-1} - \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} \\ & -(\bar{z}f_{nor}^{(\Lambda)}(z, z\bar{z}) + z\overline{f_{nor}^{(\Lambda)}(z, z\bar{z})}). \end{aligned} \tag{2.36}$$

Here we notice that

$$\begin{aligned} & \bar{z}f_{nor}^{(\Lambda)}(z,w) + z\overline{f_{nor}^{(\Lambda)}(z,w)} - (\bar{z}f_{nor}^{(\Lambda)}(z, z\bar{z}) + z\overline{f_{nor}^{(\Lambda)}(z, z\bar{z})}) \\ & \quad +\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z} + z^s)^{t-j-1} - \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} \\ & = -\bar{a}(1-s)^{j+1}z^{(j+1)s+1}z\bar{z}(z\bar{z} + z^s)^{t-j-2} + \Theta_{N_s}^{\Lambda+2} \\ & \quad +\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} + \Theta_{N_s}^{\Lambda+2} \\ & \quad +\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z} + z^s)^{t-j-1} - \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} \\ & = \bar{a}(1-s)^{j+1}z^{(j+2)s+1}(z\bar{z} + z^s)^{t-j-2} + \Theta_{N_s}^{\Lambda+2}. \end{aligned}$$

Hence we have

$$\begin{aligned} J \ &= \ (sz^{s-1} + \Theta_s^2)f_1^{(\Lambda)}(z,w) + (s\bar{z}^{s-1} + \Theta_s^2)\overline{f_1^{\Lambda}(z,w)} \\ & \quad +\bar{a}(1-s)^{j+1}z^{(j+2)s+1}(z\bar{z} + z^s)^{t-j-2} + \Theta_{N_s}^{\Lambda+2} \\ &= \ \bar{a}(1-s)^{j+2}z^{(j+2)s+1}w^{t-j-2} + \Theta_{N_s}^{\Lambda+2}. \end{aligned} \tag{2.37}$$

This proves the lemma when $m = 2t + (j+2)s + 2$. Now, the result obtained in the previous step completes the proof of the claim in this step.

Step III of the proof of Lemma 2.3: We now can complete the proof of the lemma by inductively using results obtained in Steps I-II. Indeed, since we know that the Lemma holds for $m = 2t + 2$, we see, by Step I, that the lemma holds for any $m \le N_0$ with $m \in [2t+2, 2t+(s-2)+1]$. Then, applying first Step II and then applying Step I again, we see the lemma holds for any $m \le N_0$ with $m \in [2t + j(s-2) + 2, 2t + (j+1)(s-2) + 1]$ and $j = 1$. Now, by an induction argument on j, we see the proof of the lemma. ∎

We next complete the proof of Theorem 2.2 in case $\mathrm{Ord}(f) = 2t$. First, if $m = ts + 1 < N_0$, we then have, by Lemma 2.3:

$$\begin{aligned} g_{ts+1}(w) \ &= \ \bar{a}(1-s)^t z^{ts+1} + \Theta_{ts+2}^{ts+1} + (\bar{z} + sz^{s-1} + \Theta_s^2)f_{ts}(z,w) \\ & \quad +(z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{ts}(z,w)}. \end{aligned}$$

Collecting terms of degree $ts+1$ in the above equation, we obtain:

$$g_{nor}^{(ts+1)}(z\bar{z}) = \bar{a}(1-s)^t z^{ts+1} + \mathbb{P}_{ts+2}^{ts+1} + \bar{z} f_{nor}^{(ts)}(z, z\bar{z}) + z\overline{f_{nor}^{(ts)}(z, z\bar{z})}. \tag{2.38}$$

Since $ts+2 > ts+1$, we can write $\mathbb{P}_{ts+2}^{ts+1} = \bar{z}A(z, \bar{z})$ for some polynomial function A. Hence, the equation above is solvable only if a=0, which is a contradiction.

Second, suppose $2t+1 < N_0 \leq ts+1$. By the normalization assumption in the theorem, we notice that $N_0 \neq ts+1$. Hence, we must have $2t+1 < N_0 < ts+1$

Assume that j is the integer such that $2t + j(s-2) + 2 \leq k_0 s + j_0 \leq 2t + (j+1)(s-2) + 1$. Then by Lemma 2.3 and collecting terms of degree N_0 in (2.19), we have

$$\begin{aligned} g_{nor}^{(N_0)}(z\bar{z}) &= 2Re\{(b_{N_0} - a_{N_0})z^{N_0}\} + \delta(1-s)^{j+1}\bar{a}z^{(j+1)s+1}(z\bar{z})^{t-j-1} \\ &\quad + \bar{z}f_{nor}^{(N_0)}(z, z\bar{z}) + z\overline{f_{nor}^{(N_0)}(z, z\bar{z})} + \Theta_{N_0+1}^{N_0}. \end{aligned}$$

Here $\delta = 0$ if $N_0 < 2t + (j+1)(s-2) + 1$ and $\delta = 1$ if $N_0 = 2t + (j+1)(s-2) + 1$. With the same argument above, we can see a contradiction too.

Hence, to reach no contradiction, we must have $b_N = a_N$ for any $N \leq ns + s - 1$. We thus conclude that $ts+1 \geq ns+s$ and $t \geq n+1$. This finally completes the proof.

Step II of the proof of Theorem 2.2: In this step, we show that we can also have the result stated in Theorem 2.2 when $\mathrm{Ord}(f)$ is a finite odd number by applying the same argument as in Step I.

Suppose that $\mathrm{Ord}(f) = 2t+1$, then we can still assume that $2t+2 \leq N_0$ as argued in Step I, where N_0 is defined in a similar way. Assume that $ts+s+1 < N_0$. Collecting terms of degree 2t+2 in (2.9), we get

$$g_{nor}^{(2t+2)}(z\bar{z}) = \bar{z}f_{nor}^{(2t+1)}(z, z\bar{z}) + z\overline{f_{nor}^{(2t+1)}(z, z\bar{z})}. \tag{2.39}$$

Its solution is given by

$$f_{nor}^{(2t+1)}(z, w) = bzw^t \quad , \quad g_{nor}^{(2t+2)}(w) = (b+\bar{b})w^{t+1}. \tag{2.40}$$

Substituting the solution in (2.40) to (2.39) and letting $A = (s-1)b - \bar{b}$, we get

$$\begin{aligned} g_{2t+3}(w) &= Az^s(z\bar{z} + z^s)^t + (\bar{z} + sz^{s-1} + \Theta_s^2)f_{2t+2}(z, w) \\ &\quad + (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{2t+2}(z, w)} + \Theta_{ts+s+1}^{2t+3}. \end{aligned} \tag{2.41}$$

Repeating the same induction argument as in the proof of Lemma 2.3, we get

$$\begin{aligned} g_{ts+s}(w) &= A(1-s)^t z^{ts+s} + \Theta_{ts+s+1}^{ts+s} + (\bar{z} + sz^{s-1} + \Theta_s^2)f_{ts+s-1}(z, w) \\ &\quad + (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{ts+s-1}(z, w)}. \end{aligned} \tag{2.42}$$

Collecting terms of degree ts+s in (2.42), we obtain

$$g_{nor}^{(ts+s)}(z\bar{z}) = A(1-s)^t z^{ts+s} + \mathbb{P}_{ts+s+1}^{ts+s} + \bar{z}f_{nor}^{(ts+s-1)}(z, z\bar{z}) + z\overline{f_{nor}^{(ts+s-1)}(z, z\bar{z})}. \tag{2.43}$$

As before, it is solvable only when A=0 thus b=0, which gives a contradiction. The case for $ts + s \geq N_0$ can be similarly studied to conclude that $ts + s \geq ns + s$ and thus $2t + 1 \geq 2n + 1$. This completes the proof of Theorem 2.2. ∎

3 A complete set of formal invariants, proofs of Theorem 1.1, Corollary 1.3 and Theorem 1.5

In this section, we will establish a formal normal form for the formal surface defined in (2.2), by applying a formal transformation preserving the origin. This will give a complete classification of germs of formal surfaces $(M, 0)$ with $\lambda = 0,\ s < \infty$ in the formal setting, which, in particular, can be used to answer an open question raised by J. Moser in 1985 ([pp 399, Mos]).

As another application of our complete set of formal invariants, we show that a generic Bishop surface with the Bishop invariant vanishing is not equivalent to an algebraic surface, by applying a Baire category argument similar to the study in the CR setting (see the paper of Forstneric [For]). Notice that this phenomenon is strikingly different from the theory for elliptic Bishop surfaces with non-vanishing Bishop invariants, where Moser-Webster proved their celebrated theorem, that states that any elliptic Bishop surface with a non-vanishing Bishop invariant has an algebraic normal form.

Let M be a formal Bishop surface in $\mathbb{C}^2$ defined by

$$w = H(z, \overline{z}) = z\bar{z} + 2Re\{\sum_{j=s}^{N} a_j z^j\} + E_{N+1}(z, \bar{z}), \tag{3.1}$$

where $s \geq 3$ is a positive integer and E_{N+1} is a formal power series in $(z, \overline{z})$ with $\mathrm{Ord}(E_{N+1}) \geq N + 1$. Moreover, $a_s = 1$ and for $m > s,\ m \leq N$,

$$a_m = 0 \quad \text{if} \quad m = 0, 1 \text{ mod s}.$$

Our first result of this section is the following normalization theorem:

Theorem 3.1: *With the above notation, there is a polynomial map*

$$\begin{cases} z' = z + f(z, w), & f(z, w) = O(|w| + |z|^2) \\ w' = w + g(z, w), & g(z, w) = O(|w|^2 + |z|^3 + |zw|) \end{cases} \tag{3.2}$$

that transforms the formal Bishop surface M defined in (3.1) to the formal Bishop surface defined by

$$w' = H^*(z', \overline{z'}) = z'\bar{z}' + 2Re\{\sum_{j=s}^{N+1} b_j z'^j\} + E^*_{N+2}(z', \bar{z}'). \tag{3.3}$$

Here $E^*_{N+2} = O(|z|^{N+2})$, $a_j = b_j$ *for* $s \le j \le N$ *and*

$$b_{N+1} = 0 \quad if \quad N+1 = 0, 1 \ mod \ s.$$

Moreover, when $N+1 \ne 0,1 \ \ mod(s)$, $wt_{nor}(f) \ge N$ *and* $wt_{nor}(g) \ge N+1$*; when* $N = ts$, $wt_{nor}(f) \ge 2t, \ \ wt_{nor}(g) \ge 2t+1$. *and when* $N = ts-1$, $wt_{nor}(f) \ge 2t-1, \ \ wt_{nor}(g) \ge 2t$.

Before proceeding to the proof, we recall a result of Moser, which will be used for our consideration here. For any $m \ge 4$ and holomorphic polynomials

$$f^{(m-1)}_{nor}(z,w), \ g^{(m)}_{nor}(z,w), \ \phi^{(m)}(z),$$

we define an operator, which we call the Moser operator $\mathcal{L}$, as follows:

$$\mathcal{L}(f^{(m-1)}_{nor}(z,w), g^{(m)}_{nor}(z,w), \phi^{(m)}(z)) := g^{(m)}_{nor}(z, z\overline{z}) - 2Re\{\overline{z} f^{(m-1)}_{nor}(z, z\overline{z}) + \phi^{(m)}(z)\}.$$

The following lemma is essentially the content of Proposition 2.1 of [Mos]:

Lemma 3.2: Let $G(z,\overline{z})$ be a homogeneous polynomial of degree m. Then

$$\mathcal{L}(f^{(m-1)}_{nor}(z,w), g^{(m)}_{nor}(z,w), \phi^{(m)}(z)) = G(z,\overline{z})$$

has a unique solution: $\{f^{(m-1)}_{nor}(z,w), g^{(m)}_{nor}(z,w), \phi^{(m)}\}$ under the normalization condition: $f^{(m-1)}_{nor} = z^2 f^*$ with f^* a holomorphic polynomial. Moreover, when G has no harmonic terms, then $\mathcal{L}\left(f^{(m-1)}_{nor}(z,w), g^{(m)}_{nor}(z,w), 0\right) = G(z,\overline{z})$ also has a unique solution $\{f^{(m-1)}_{nor}(z,w), g^{(m)}_{nor}(z,w)\}$ under the same normalization condition just mentioned.

The proof of the Theorem 3.1 follows from a similar induction argument that we used in the previous section.

Proof of Theorem 3.1: We complete the proof in three steps.

Step 1: We first show that there is a polynomial map: $z' = z + f^{(N)}_{nor}(z,w)$, $w' = w + g^{(N+1)}_{nor}(z,w)$, which maps M to a surface defined by the following equation:

$$w = z\bar{z} + 2Re\{\sum_{j=s}^{N+1} b_j z^j\} + E^*_{N+2}(z,\bar{z}) \tag{3.4}$$

with $b_j = a_j$ for $s \le j \le N$. Substituting the map into (3.4) and collecting terms of degree $N+1$, we see that the existence of the map is equivalent to the existence of solutions of the following functional equation:

$$\mathcal{L}(f^{(N)}_{nor}(z,w), g^{(N+1)}_{nor}(z,w), b_{N+1}z^{N+1}) = -E^{(N+1)}_{N+1}(z,\overline{z}). \tag{3.5}$$

By Lemma 3.2, we know that (3.5) is indeed solvable and is uniquely solvable under the normalization condition as in Lemma 3.2.

For the rest of the proof of the theorem, we can assume that $E_{N+1} = 2Re\{b_{N+1}z^{N+1}\} + o(|z|^{N+1})$.

Step 2: In this step, we assume that $N+1 = 1$ mod s. Write $N = ts$. We then show that there is a polynomial map of the form:

$$\begin{aligned} z' &= z + \textstyle\sum_{l=0}^{N-2t}\{f^{(2t+l)}(z,w)\}, \\ w' &= w + \textstyle\sum_{\tau=0}^{N+1-2t-2}\{g_{nor}^{(2t+2+\tau)}(w)\} \end{aligned} \tag{3.6}$$

such that under this transformation, M is mapped to a formal surface M' defined by (3.3) with $b_{N+1} = 0$. The map is also uniquely determined by imposing the normalization condition as in Lemma 3.2 for $f^{(j)}$ with $2t < j \le N+1$.

As in Step I, this amounts to studying a series of normally weighted homogeneous functional equations with the normally weighted degree running from $2t$ to $N+1$. Substituting (3.6) into (3.3) and then collecting terms of degree $2t+1$, we obtain the equation (2.12), which can be solved as:

$$f_{nor}^{(2t)}(z,w) = aw^t - \bar{a}z^2w^{t-1}$$

with a to be (uniquely) determined later.

Now, suppose we are able to solve $f_{nor}^{(2t+l)},\ g_{nor}^{(2t+1+l)}$ for $2t+l = 2t, \cdots, m-1 \le st-2$. Substituting (3.6) into (3.3) and then collecting terms of degree $m+1$, we obtain an equation similar to (2.20), as argued in the proof of Lemma 2.3:

$$g^{(m+1)}(z\bar{z}) = \bar{z}f_{nor}^{(m)}(z,z\bar{z}) + z\overline{f_{nor}^{(m)}(z,z\bar{z})} + \hat{\mathbb{P}}_{ts+2}^{m+1} \tag{3.7}$$

Notice that $\hat{\mathbb{P}}_{ts+2}^{m+1} (= \mathbb{P}_{ts+2}^{m+1})$ must be real valued and is uniquely determined by the known data. This equation, in terms of the Moser operator, can be rewritten as:

$$\mathcal{L}\left(f_{nor}^{(m)}(z,z\bar{z}), g^{(m+1)}(z\bar{z}), 0\right) = \hat{\mathbb{P}}_{ts+2}^{m+1}. \tag{3.8}$$

Since $\hat{\mathbb{P}}_{ts+2}^{m+1}$ is real-valued and divisible by $\overline{z}$, it does not contain any harmonic terms. By Lemma 3.2, it can be solved, and can be uniquely solved under the normalization condition in Lemma 3.2. By induction, we can uniquely obtain $f_{nor}^{(m)},\ g_{nor}^{(m+1)}$ for $m \le ts-1$. Substituting (3.6) into (3.3) and then collecting terms of degree $m = ts+1$, we obtain an equation similar to (2.38), which can be rewritten as:

$$\begin{aligned} \mathcal{L}(g_{nor}^{(ts+1)}(z\bar{z}), f_{nor}^{(ts)}(z,z\overline{z}), 0) &= 2Re\{\bar{a}(1-s)^t z^{ts+1}\} \\ &+ \hat{\mathbb{P}}_{ts+2}^{ts+1} - a(1-s)^t\overline{z}^{ts+1} - 2Re(b_{ts+1}z^{ts+1}). \end{aligned} \tag{3.9}$$

As in the proof of Theorem 2.2, the real-valued homogeneous polynomial $\hat{\mathbb{P}_{ts+2}^{ts+1}} - a(1-s)^t\overline{z}^{ts+1}$ has a $\overline{z}$ factor and thus has no harmonic terms. Hence, if we choose $a = \overline{b_{ts+1}}/(1-s)^t$, then

(3.9) is uniquely solvable, under the normalization condition in Lemma 3.2. This completes the proof of the claim in this step.

Step 3: In this step, we assume that $N+1=0$ mod s. Write $N=(t+1)s-1$. We then show that there is a unique polynomial map of the form:

$$\begin{aligned} z' &= z+\sum_{l=0}^{N-1-2t}\{f_{nor}^{(2t+l+1)}(z,w)\}, \\ w' &= w+\sum_{\tau=0}^{N+1-2t-2}\{g_{nor}^{(2t+2+\tau)}(w)\} \end{aligned} \tag{3.10}$$

such that under this transformation, M is mapped to a formal surface M' defined by (3.3) with $b_{N+1}=0$. Here $f_{nor}^{(m)}$ satisfies the normalization condition in Lemma 3.2 for $m\neq 2t+1$.

The argument for this step is the same as that for Step 2. We first have to choose

$$f_{nor}^{(2t+1)}(z,w)=bzw^t, \qquad g_{nor}^{(2t+2)}(w)=(b+\bar{b})w^{t+1}$$

with b to be uniquely determined later. Arguing exactly in the same way as in Step 2, we can inductively find the unique solution (under the normalization condition) for $f_{nor}^{(2t+l)}$, $g_{nor}^{(2t+1+l)}$ with $2t+l=2t+2,\cdots,<st+s-1$. At the level with degree $ts+s$, we have the following equation:

$$\begin{aligned} 2Re(b_{N+1}z^{N+1})+g_{nor}^{(ts+s)}(z\bar{z}) &= ((s-1)b-\bar{b})(1-s)^t z^{ts+s} \\ &+\hat{\mathbb{P}}_{ts+s+1}^{ts+s}+\bar{z}f^{(ts+s-1)}(z,z\bar{z})+z\overline{f^{(ts+s-1)}(z,z\bar{z})}. \end{aligned} \tag{3.11}$$

Now, arguing the same way as in Step 2, the equation (3.11) is uniquely solvable by taking b such that $(s-1)b-\overline{b}=b_{N+1}$ and by imposing the normalization condition as in Lemma 3.2 to $f_{nor}^{(ts+s-1)}$.

Now, the map in Theorem 3.1 can be chosen as the map in Step 1 if $N+1\neq 0,1 \bmod(s)$. When $N+1=0$, or $1 \bmod(s)$, the map in Theorem 3.1 can be defined by composing the map in Step 2 or that in Step 3, respectively, with the map in Step 1. We see the proof of Theorem 3.1. Moreover, with such fixed procedures and normalizations described in the above steps, there are a set of universal polynomials $\{P_{kl}(a_{\alpha\beta})\}_{1\leq\alpha+\beta\leq k+l}$ (depending only on s and N) such that the coefficients of the map $(z',w')=(z,w)+(f,g)=(z,w)+\sum_{k,l}b_{kl}z^k w^l$ in Theorem 3.1 are determined by

$$b_{kl}=P_{kl}(a_{\alpha\beta}), \quad 1\leq\alpha+\beta\leq k+l \tag{3.12}$$

where $H=\sum_{\alpha,\beta\geq 0}a_{\alpha\beta}z^\alpha\overline{z}^\beta$.

The last sentence in Theorem 3.1 follows from the procedures that we used to prove the existence part. ■

We next choose the map $z'=z+f,\ w'=w+g$ in Theorem 3.1 such that its coefficients are determined by (3.12). Let $z=z'+f^*(z',w')$ and $w=w'+g^*(z',w')$ be its inverse transformation. Notice that the coefficients of (f^*,g^*) in its Taylor expansion up to degree, say

m, are universal polynomial functions of the coefficients of (f, g) up to degree m for any m. Hence we have the defining equation of M^*, the image of M, as follows:

$$w' + g^*(z', w') = H(z' + f^*(z', w'), \overline{z' + f^*(z', w')}).$$

Applying an implicit function theorem to solve for w' and making use of the uniqueness of the graph function, we see that the coefficients in the Taylor expansion of H^* up to degree m must also be polynomial functions of the coefficients of H of degree not exceeding m in its Taylor expansion. Repeating such a normalization procedure that we did for M to M^* and by an induction argument, we get the following theorem: (The uniqueness part follows from Theorem 2.2.)

Theorem 3.3: *Let M be a formal Bishop surface defined by*

$$w = H(z, \overline{z}) = z\bar{z} + z^s + \bar{z}^s + E(z, \bar{z}), \tag{3.13}$$

where $s \geq 3$ is a positive integer and $E(z, \overline{z}) = \sum_{\alpha+\beta\geq s+1}^{\infty} a_{\alpha\beta} z^{\alpha} \overline{z^{\beta}}$. Then there is a unique formal transformation of the form:

$$\begin{cases} z' = z + f(z, w), & f(z, w) = O(|w| + |z|^2) \\ w' = w + g(z, w), & g(z, w) = O(|w|^2 + |z|^3 + |zw|) \end{cases} \tag{3.14}$$

that transforms M to the formal Bishop surface defined by

$$w' = H^*(z', \overline{z'}) = z'\bar{z}' + z'^s + \overline{z'}^s + 2Re\{\sum_{j=2,\cdots,s-1;\ k\geq 1}^{\infty} \lambda_{ks+j} z'^{ks+j}\}. \tag{3.15}$$

The normal form in (3.15), up to a transformation of the form $z'' = e^{i\theta} z',\ w'' = w$ with $e^{is\theta} = 1$, uniquely determines the formal equivalence class of M. Moreover, there are a set of universal polynomial functions

$$\{\Lambda_{ks+j}(Z_{\alpha\beta})\}_{s+1\leq\alpha+\beta\leq ks+j;\ j=2,\cdots,s-1;\ k\geq 1}$$

depending only on s, such that:

$$\lambda_{ks+j} = \Lambda_{ks+j}(a_{\alpha\beta})_{s+1\leq\alpha+\beta\leq ks+j;\ j=2,\cdots,s-1;\ k\geq 1}. \tag{3.16}$$

Proofs of Theorem 1.1 and Corollary 1.3: Theorem 1.1 follows immediately from Theorem 3.3 and Lemma 2.1 (ii).

The proof of Corollary 1.3 (a), (b), (d) also follows easily from Theorem 3.1. To see Corollary 1.3 (c), we let $\mathcal{G}$ be a proper subgroup of $\mathcal{Z}_s$. Define $J_G := \{j : \ 2 \leq j \leq s-1,\ e^{i\theta j} = 1,$ for any $(e^{i\theta} z, w) \in \mathcal{G}\}$. Let M_G be defined by

$$w = z\overline{z} + z^s + \overline{z}^s + 2Re\{\sum_{j\in J_G} a_{s+j} z^{s+j}\},$$

with $a_{s+j} \neq 0$. Then we will verify that $aut_0(M_G) = \mathcal{G}$. To this aim, write $\mathcal{G}^*$ to be the collection of $\xi's$ with $(z, w) \to (\xi z, w)$ belonging to $\mathcal{G}$. By Corollary 1.3 (a), we need only to show that if $\xi^{*s} = 1$ and $\xi^{*j} = 1$ for any $j \in J_G$, then $\xi^* \in \mathcal{G}^*$. Write $k = |\mathcal{G}^*|$. Then $s = km$ with $m(\in \mathbf{N}) > 1$. For any $\xi(\in \mathcal{G}^*) \neq 1$, since the order of ξ must be divisible by k, we see that $\xi^k = 1$. Therefore, $\mathcal{G}^*$ forms a complete set of the solutions of $z^k = 1$. Now, it is clear that $J_G = \{k, \cdots, (m-1)k\}$. Hence, we see that $\xi^{*k} = 1$. Thus, $\xi^* \in \mathcal{G}^*$. This completes the proof of Corollary 1.3 (c).

Now, by Corollary 1.3 (a), we see that for M as in Corollary 1.3 (e), M must be formally equivalent to M_s. Assuming Theorem 1.2, which we will prove in the next section, we also conclude that M is biholomorphically equivalent to M_s. Corollary 1.3 (f) is a simple consequence of the results in (a) and (e). ∎

Corollary 3.4: *Let M be a real analytic Bishop surface defined by an equation of the form:*

$$w = H(z, \overline{z}) = z\overline{z} + 2Re\{z^s + \sum_{k\geq 1,\ j=2,\cdots,s-1} a_{ks+j} z^{ks+j}\} \quad \text{with infinitely many} \quad a_{ks+j} \neq 0.$$

Then for any $N > s$, M is not equivalent to the Bishop surface M_N defined by

$$w = H_{(N+1)}(z, \overline{z}) = z\overline{z} + 2Re\{z^s + \sum_{k\geq 1,\ j=2,\cdots,s-1}^{ks+j\leq N} a_{ks+j} z^{ks+j}\}.$$

Here $H_{(N+1)}$ is the N^{th}-truncation from the Taylor expansion of H at 0. In fact, $M_{(N+1)}$ is equivalent to $M_{(N'+1)}$ with $N' > N$ if and only if $a_{ks+j} = 0$ for any $N < ks + j \leq N'$.

Corollary 3.4 answers, in the negative, the second problem that J. Moser asked in his paper ([pp 399, Mos]).

As a less obvious application of Theorem 3.3, we next show that a generic Bishop surface with the Bishop invariant vanishing at 0 and with $s < \infty$ is not even formally equivalent to any algebraic surface in $\mathbb{C}^2$. For this purpose, we borrow the idea used in the CR setting based on the Baire category argument. For the consideration in the CR setting by using the Baire category theorem, the reader is referred to the paper of Forstneric [For].

Write $\mathcal{M}_s$ for the collection of all formal Bishop surfaces defined as in (3.13):

$$w = H(z, \overline{z}) = z\overline{z} + 2Re(z^s) + \sum_{\alpha+\beta\geq s+1} a_{\alpha\beta} z^\alpha \overline{z^\beta}. \tag{3.17}$$

Write $\mathcal{F} := \{\vec{a} = (a_1, \cdots, a_n, \cdots) : \ a_j \in \mathbb{C}\}$, equipped with the usual distance function:

$$dist(\vec{a}, \vec{b}) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{2^j(1 + |a_j - b_j|)}.$$

We know that $\mathcal{F}$ is a Frèchet space. There is a one-to-one correspondence between $\mathcal{M}_s$ and $\mathcal{F}$, which assigns each $M \in \mathcal{M}_s$ to an element: $\vec{M} = (a_{\alpha\beta}) \in \mathcal{F}$ labeled in the lexicographical order. Therefore, we can, in what follows, identify $\mathcal{M}_s$ as a Frèchet space. We define the operator $\mathcal{J}$ such that it sends any $M \in \mathcal{M}_s$ to $(\lambda_{ks+j})_{j\neq 0,1;k\geq 1}$, where (λ_{sk+j}) is described as in Theorem 3.3. By (3.16), we easily see that $\mathcal{J}$ is a continuous map from $\mathcal{M}_s$ to $\mathcal{F}$.

(M,p) in $\mathbb{C}^2$ is called the germ of an algebraic surface if M near p possesses a real polynomial defining equation. If $p \in M$ is a point with an elliptic complex tangent, whose Bishop invariant is 0 and whose Moser invariant is $s < \infty$, then there is a change of coordinates (see [Hu1], for instance) such that $p = 0$ and M near 0 is defined by an equation of the form:

$$w = z\overline{z} + B(z,\overline{z},w,\overline{w}), \quad B(z,\overline{z},w,\overline{w}) = \sum_{3\leq \alpha+\beta+2\gamma+2\tau} c_{\alpha\beta\gamma\tau} z^{\alpha}\overline{z}^{\beta}w^{\gamma}\overline{w}^{\tau}, \tag{3.18}$$

where B is a polynomial in its variables. By using the implicit function theorem and using the argument in the step 1 of the proof of Theorem 3.1, it is not hard to see that there is a fixed procedure to transform (3.18) into a surface defined by an equation as in (3.17), in which $a_{\alpha\beta}$ are presented by polynomials of $c_{\alpha\beta\gamma\tau}$ and $H(z,\overline{z})$ becomes what we call a Nash algebraic function to be defined as follows:

We call a real analytic function $h(z,\overline{z})$ near 0 a Nash algebraic function if either $h \equiv 0$ or there is an irreducible polynomial $P(z,\overline{z};X)$ in X with polynomial coefficients in $(z,\overline{z})$ such that $P(z,\overline{z};h(z,\overline{z})) \equiv 0$. Certainly, we can always assume that the coefficients of (z,ξ,X) (in $P(z,\xi,X)$) of terms with highest power in X have maximum value 1. The degree of h is defined as the total degree of P in $(z,\overline{z},X)$.

For $d,\ n,\ m \geq 1$, we define $\mathcal{A}^d_B(n,m) \subset \mathcal{M}_s$ to be the subset of Bishop surfaces defined in (3.17), where $H(z,\overline{z})'s$ are Nash algebraic functions derived from the $B's$ in (3.18) in the procedure described above with the degree of $B's$ bounded by d, that further satisfy the following properties:

Cond (1): $H(z,\xi)'s$ are holomorphic over $|z|^2 + |\xi|^2 < 1/m^2$;

Cond(2): $\max_{(|z|^2+|\xi|^2)<1/m^2} |H(z,\xi)| \leq n$ and $|c_{\alpha\beta\gamma\tau}| \leq n$.

Write $\mathcal{A}^d_B = \cup_{n,m=1}^{\infty}\mathcal{A}^d_B(n,m)$ and $\mathcal{A}_B = \cup_{d=1}^{\infty}\mathcal{A}^d_B$. It is a consequence of Theorem 3.3 that M, defined in (3.13), is formally equivalent to an algebraic surface if and only if $\mathcal{J}(M) \in \mathcal{J}(\mathcal{A}_B)$. (Therefore, M defined in (3.13) is not formally equivalent to an algebraic surface if and only if $\mathcal{J}(M) \notin \mathcal{J}(\mathcal{A}_B)$.)

Now, for any sequence $\{M_j\} \subset \mathcal{A}^d_B(n,m)$ with $M_j : w = H_j(z,\overline{z}) = z\overline{z} + z^s + \overline{z}^s + o(|z|^s)$, by a normal family argument and by passing to a subsequence, we can assume that $H_j(z,\xi) \to H_0(z,\overline{z})$ over any compact subset of $\{|z|^2 + |\xi|^2 < 1/m^2\}$. If follows easily that M_0 defined by $w = H_0$ is also in $\mathcal{A}^d_B(n,m)$. Moreover, $D^{\alpha}_z D^{\beta}_{\xi} H_j(0) \to D^{\alpha}_z D^{\beta}_{\xi} H_0(0)$ for any (α,β). By (3.16), $\mathcal{J}(M_j) \to \mathcal{J}(M_0)$ in the topology of $\mathcal{F}$. Therefore, we easily see that $\mathcal{J}(\mathcal{A}_B)$ is a subset of $\mathcal{F}$ of the first category.

Next, for any $R > 0$, we let

$$\mathcal{S}_R := \{\vec{\lambda} = (\lambda_{sk+j})_{k\geq 1; j=2,\cdots,s-1}\} : \ \|\vec{\lambda}\|_R := \sum_{ks+j} |\lambda_{ks+j}| R^{ks+j} < \infty\}.$$

It can be verified that $\mathcal{S}_R$ is a Banach space under the above defined $\|\cdot\|_R$-norm. (In fact, it reduces to the standard l^1-space when $R = 1$.) We now claim that $\mathcal{K}_B^d$, defined as the closure of $\mathcal{J}\left(\mathcal{A}_B^d(n,m)\right) \cap \mathcal{S}_R$ in $\mathcal{S}_R$ in its Banach norm, has no interior point.

Suppose, to the contrary, that a certain ϵ-ball $\mathcal{B}$ of $\vec{a_0} = (\lambda_{sk+j}^0)_{k\geq 1; j=2,\cdots,s-1}$ in $\mathcal{S}_R$ is contained in $\mathcal{K}_B^d$. We must then have $\mathcal{B} \subset \mathcal{J}\left(\mathcal{A}_B^d(n,m)\right) \cap \mathcal{S}_R$. Indeed, for any $\vec{a} \in \mathcal{B}$, let $\mathcal{J}(M_j) \to \vec{a}$ with $M_j \in \mathcal{A}_B^d(n,m)$. By the argument in the above paragraph, we can assume, without loss of generality, that $M_j \to M_0 \in \mathcal{A}_B^d(n,m)$ in the $\mathcal{F}$-norm. By (3.16), we see that $\mathcal{J}(M_0) = \vec{a}$. Choose $\vec{a} = \{\lambda_{ks+j}\}$ such that $|\lambda_{ks+j} - \lambda_{ks+j}^0| \cdot (2R)^{ks+j} < \epsilon$ for any $ks+j$. For any $N \geq 1$, then we see that there is a certain $H = z\overline{z} + z^s + \overline{z^s} + \sum_{s+1\leq \alpha+\beta} a_{\alpha\beta} z^\alpha \overline{z^\beta}$ Nash algebraic near 0 such that

$$\lambda_{ks+j} = \Lambda_{ks+j}(a_{\alpha\beta}), \quad N \geq ks+j \geq s+1, \ \alpha+\beta \leq ks+j, \quad \Lambda = (\Lambda_{ks+j})_{s+1\leq ks+j\leq N}. \tag{3.19}$$

Here H is obtained from B in (3.18) with degree of B bounded by d. Since $a_{\alpha\beta}$ are polynomial functions of $c_{\alpha\beta\gamma\tau}$, we can conclude a contradiction from (3.19). Indeed, since the variables on the right hand side of (3.19) are polynomially parametrized by less than d^4 free variables $(c_{\alpha\beta\gamma\tau})$, the image of (3.19) can not fill in an open subset of $\mathbb{R}^{N-s}$ as $N >> 1$.

Therefore, we proved that $\mathcal{A}_B = \cup_{d,n,m=1}^{\infty} \mathcal{A}_B^d(n,m)$ is a set of the first category in $\mathcal{S}_R$. By the Baire category theorem, we conclude that most elements in $\mathcal{S}_R$ are not from $\mathcal{J}\left(\mathcal{A}_B \cap \mathcal{S}_R\right)$. For any $\vec{a} = (\lambda_{sk+j}) \notin \mathcal{J}\left(\mathcal{A}_B \cap \mathcal{S}_R\right)$, the Bishop surface defined by: $w = z\overline{z} + z^s + \overline{z}^s + 2\mathrm{Re}(\sum_{k\geq 1; j\neq 0,1} \lambda_{ks+j} z^{ks+j})$ is not equivalent to any algebraic surface in $\mathbb{C}^2$. When R varies, we complete a proof of Theorem 1.5. ■

A real analytic surface in $\mathbb{C}^2$ is called a Nash algebraic surface if it can be defined by a Nash algebraic function. By the same token, we can similarly prove the following:

Theorem 3.5: Most real analytic elliptic Bishop surfaces with the Bishop invariant $\lambda = 0$ and the Moser invariant $s < \infty$ at 0 are not equivalent to Nash algebraic surfaces in $\mathbb{C}^2$.

Proof of Theorem 3.5: To prove Theorem 3.5, we define $\mathcal{A}_B^d(n,m)$ in the same way as before except that we now only require that $H(z,\overline{z}) = z\overline{z} + z^s + \overline{z}^s + \sum_{\alpha+\beta\geq s+1} a_{\alpha\beta} z^\alpha \overline{z}^\beta$ is a general Nash algebraic function with total degree bounded by d and with the same conditions described as in Cond (1) and the first part of Cond (2). The last part of Cond (2) is replaced by the condition that $|b_{\alpha\beta\gamma}| \leq n$, where $P(z,\overline{z},X) = \sum b_j(z,\overline{z}) X^j = \sum_{\alpha\beta\gamma} b_{\alpha\beta\gamma} z^\alpha \overline{z}^\beta X^\gamma$ is a minimal polynomial of H with the same coefficient restriction as imposed before.

We fix an H_0 and its minimal polynomial $P_0(z,\overline{z};X)$. (We will fix certain coefficient of P in the top degree terms of X to be 1 to make the minimal polynomial P_0 unique). Let

$\mathcal{A}_B^d(n,m;H_0,\delta)$ be a subset of $\mathcal{A}_B^d(n,m)$, where $M=\{w=H(z,\overline{z})\}\in\mathcal{A}_B^d(n,m;H_0,\delta)$ if and only if $|b_{\alpha\beta\gamma}-b^0_{\alpha\beta\gamma}|\leq\delta$. Here $P=\sum b_{\alpha\beta\gamma}z^\alpha\overline{z}^\beta X^\gamma$ and $P_0=\sum b^0_{\alpha\beta\gamma}z^\alpha\overline{z}^\beta X^\gamma$ are the minimal polynomials of H and H_0, respectively. We assume that P is normalized in the same manner as for P_0. (Certainly, we can always do this if $\delta<<1$.)

Consider an H and its minimal polynomial P associated with an element from $\mathcal{A}_B^d(n,m;H_0,\delta)$. Let R be the resultant of P and P'_X with respect to X. We know that R is a non-zero polynomial of $(z,\overline{z})$ of degree bounded by $C_1(d)$, a constant depending only on d. Write $H=H^*_{(N)}+H^{**}_N$ with $H^*_{(N)}$ the Taylor polynomial of H up to order $N-1$ and H^{**}_N the remainder. Then from $P(z,\overline{z},H^*_{(N)}+H^{**}_N)=0$, we obtain

$$P^{**}(z,\overline{z},X^{**})=0 \quad \text{with} \quad X^{**}=H^{**}_N. \tag{3.20}$$

Here P^{**} is a polynomial of total degree bounded by $C_2(d,N)$, a constant depending only on d and N, and its coefficients are determined polynomially by the coefficients of P and $H^*_{(N)}$. Notice that $D_{X^{**}}\left(P^{**}(z,\overline{z},X^{**})\right)|_{X^{**}=0}=D_X\left(P(z,\overline{z},X)\right)_{X=H^*_{(N)}}$. Since there are polynomials G_1 and G_2 such that $G_1P+G_2P'_X=R$ and since $P(z,\overline{z},H^*_{(N)})=o(|z|^N)$, we conclude that the degree k_0 of the lowest non-vanishing order term of $P'_X(z,\overline{z},H^*_{(N)})$ is bounded by $C_1(d)$, depending only on d.

Choose an $N\geq C_1(d)$ and a sufficiently small positive number δ. We can apply a comparing coefficient method to (3.20) to conclude that each $a_{\alpha_0\beta_0}$ for $\alpha_0+\beta_0\geq N$ is determined by $b_{\alpha\beta\gamma}$ and $a_{\alpha\beta}$ with $\alpha+\beta\leq N-1$ through at most $C(k_0,N)$ rational functions in $b_{\alpha\beta\gamma}$ and $a_{\alpha\beta}$ $(\alpha+\beta\leq N-1)$ with $C(k_0,N)$ depending only on k_0,N. Now, (3.19) can be used in the same manner to show that the interior of the closure of $\mathcal{J}(\mathcal{A}_B^d(n,m;H_0,\delta))\cap\mathcal{S}_R$ in $\mathcal{S}_R$ is empty. It is easy to see that $\mathcal{J}(\mathcal{A}_B)$ can be written as a countable union of these sets. We see that $\mathcal{J}(\mathcal{A}_B)$ is a set of the first category in $\mathcal{S}_R$. This completes the proof of Theorem 3.5. ∎

Remark 3.6 (A): The crucial point for Theorem 3.5 to hold is that the modular space of surfaces with a vanishing Bishop invariant and $s<\infty$ is parameterized by an infinitely dimensional space. Hence, any subclass of $\mathcal{M}_s$, that is represented by a countable union of finite dimensional subspaces of $\mathcal{M}_s$, is a thin set of $\mathcal{M}_s$ under the equivalence relation. This idea, that the infinite dimensionality of the modular space would generally have the consequence of the generic non-algebraicity for its elements, dates back to the early work of Poincaré [Po]. In the CR setting, Forstneric in [Fo] has used the infinitely dimensional modular space of CR manifolds and the Baire category argument to give a short and quick proof that a generic CR submanifold in a complex space is not holomorphically equivalent to any algebraic manifold. Some earlier studies related to non-algebraicity for CR manifolds can be found, for instance, in [BER] [Hu2] [Ji]. However, by a result of the first author with Krantz [HK] and a result of the first author in [Hu3], a Bishop surface with an elliptic complex tangent can always be holomoirphically transformed into the algebraic Levi-flat hypersurface $\mathbb{C}\times\mathbb{R}$ and also into the Heisenberg hypersurface in $\mathbb{C}^2$.

(B). In the normal form (3.15), the condition that $\lambda_{ks+j}=0$ for $j=0,1,\ k=1,2,\cdots$ can be compared with the Cartan-Chern-Moser chain condition in the case of strongly pseudoconvex

hypersurfaces (see [CM]). In the hypersurface case, the chain condition is also described by a finite system of differential equations. It would be very interesting to know if, in our setting here, there also exist a finite set of differential equations describing our chain condition.

4 Surface hyperbolic geometry and a convergence argument

In this section, we study the convergence problem for the formal consideration in the previous section. Our starting point is the flattening theorem of Huang-Krantz [HK], which says that an elliptic Bishop surface with a vanishing Bishop invariant can be holomorphically mapped to $\mathbb{C} \times \mathbb{R}$.

Hence, to study the convergence problem, we can restrict ourselves to a real analytic Bishop surface M defined by

$$w = z\bar{z} + z^s + \bar{z}^s + E(z,\bar{z}),\ E(z,\bar{z}) = \overline{E(z,\bar{z})} = o(|z|^s),\ 3 \leq s < \infty. \tag{4.1}$$

Recall that the Moser-Webster complexification $\mathfrak{M}$ of M is the complex surface near $0 \in \mathbb{C}^4$ defined by:

$$\begin{cases} w = z\zeta + z^s + \zeta^s + E(z,\zeta) \\ \eta = z\zeta + z^s + \zeta^s + E(z,\zeta). \end{cases} \tag{4.2}$$

We define the projection $\pi : \mathfrak{M} \longrightarrow \mathbb{C}^2$ by sending $(z,\zeta,w,\eta) \in \mathfrak{M}$ to (z,w). Then π is generically s to 1. Write B for the branching locus of π. Namely, $(z,w) \in B$ if and only if $\exists(\zeta_0,\eta_0)$ such that $(z,\zeta_0,w,\eta_0) \in \mathfrak{M}$ and π is not biholomorphic near (z,ζ_0,w,η_0). Write $\mathfrak{B} = \pi^{-1}(B)$.

Then

$$\begin{aligned} &(z,w) \in B \\ &\Longleftrightarrow \exists\zeta \ \text{ such that } \ w = z\zeta + z^s + \zeta^s + E(z,\zeta) \ \text{ and } \ z + s\zeta^{s-1} + E_\zeta(z,\zeta) = 0 \\ &\Longleftrightarrow \sharp\{\pi^{-1}(z,w)\} < s. \end{aligned}$$

It is easy to see that near 0, B is a holomorphic curve passing through the origin.

Now, suppose M' is defined by $w' = z'\bar{z'} + z'^s + \bar{z'}^s + E^*(z',\bar{z'})$ with $E^*(z',\bar{z'}) = \overline{E^*(z',\bar{z'})}$ near 0. Write $\mathfrak{M}'$ for the complexification of M'. Suppose that $F : (M,0) \longrightarrow (M',0)$ is a biholomorphic map. Then F induces a biholomorphic map $\mathcal{F}$ from $(\mathfrak{M},0)$ to $(\mathfrak{M}',0)$ such that $\pi' \circ \mathcal{F} = F \circ \pi$. From this, it follows that $F(B) = B'$, where B' is the branching locus of π' near the origin.

We next give the precise defining equation of B near 0. From the equation $z + s\zeta^{s-1} + E_\zeta(z,\zeta) = 0$, we can solve, by the implicit function theorem, that

$$z = h_1(\zeta) = -s\zeta^{s-1} + o(\zeta^{s-1}), \tag{4.3}$$

where $h_1(\zeta)$ is holomorphic near 0. Substituting (4.3) into (4.2), we get

$$w = h_2(\zeta) = (1-s)\zeta^s + o(\zeta^s). \tag{4.4}$$

From (4.4), we get

$$-\frac{w}{s-1} = (h_3(\zeta))^s \quad \text{with} \quad h_3(\zeta) = \zeta + o(\zeta).$$

Hence, we get

$$\begin{aligned} \zeta &= h_3^{-1}((-\tfrac{w}{s-1})^{\frac{1}{s}}) = (-1)^{\frac{1}{s}}(\tfrac{1}{s-1})^{\frac{1}{s}}w^{\frac{1}{s}} + o(w^{\frac{1}{s}}) \\ z &= h_1((-1)^{\frac{1}{s}}(\tfrac{w}{s-1})^{\frac{1}{s}} + o(w^{\frac{1}{s}})) = s(-1)^{-\frac{1}{s}}w^{\frac{s-1}{s}} \cdot (s-1)^{\frac{1-s}{s}} + o(w^{\frac{s-1}{s}}). \end{aligned} \tag{4.5}$$

Here, $h_j's$ are holomorphic functions near 0. Next, let $w = u \geq 0$ and write

$$\begin{aligned} A_j(u) &= h_1 \circ h_3^{-1}\left(e^{-\frac{(2j+1)\pi\sqrt{-1}}{s}}(\tfrac{u}{s-1})^{1/s}\right) \\ &= se^{\frac{(1+2j)\pi\sqrt{-1}}{s}}u^{\frac{s-1}{s}} \cdot (s-1)^{\frac{1-s}{s}} + o(u^{\frac{s-1}{s}}), \quad j = 0, 1, \cdots, s-1. \end{aligned} \tag{4.6}$$

Lemma 4.1: For $0 < u \ll 1$, $A_j(u) \in D(u)$. Here

$$D(u) = \{z \in \mathbb{C}^1 : w = z\bar{z} + z^s + \bar{z}^s + E(z,\bar{z}) < u\}.$$

Proof of Lemma 4.1: The proof follows clearly from the following estimate:

$$|A_j(u)|^2 + Re\{2A_j^s(u) + E(A_j(u), \overline{A_j(u)})\} = O(u^{\frac{2(s-1)}{s}}) \ll u$$

as far as $0 < u \ll 1$ and $s \geq 3$. ∎

The following fact will be crucial for our later discussions:

$\{(A_j(u), u)\}_{j=0}^{s-1} = B \cap \{w = u\}$ and $A_j(u)$ is real analytic in $u^{1/s}$ for each fixed j.

Consider a surface (M,p) in $\mathbb{C}^2$. We say that M near p is defined by a complex-valued function ρ, if M near p is precisely the zero set of ρ and $\{Re(\rho), Im(\rho)\}$ has constant rank two near p as functions in (x,y,u,v). For a surface (M,p) defined by ρ and a biholomorphic map F from a neigborhood of p to a neighborhood of p', we say that $F(M)$ approximates (M^*, p') defined by $\rho^* = 0$ to the order m at p' if there are smooth functions h_1 and h_2 with $|h_1|^2 - |h_2|^2 \neq 0$ at p' such that $\rho \circ F^{-1}(Z) = h_1 \cdot \rho^* + h_2 \cdot \overline{\rho^*} + o(|Z - p'|^m)$.

Lemma 4.2: Let $M,\ M'$ be Bishop surfaces near 0 as defined above. Suppose that $F(M)$ approximates M' to the order $\widetilde{N} = Ns + s - 1$ at 0 with $N > 1$. Then

$$|F(A_j(u), u) - (A'_j(u'), u')| \lesssim |u|^N, \text{ for } j = 0, \cdots, s-1,\ u > 0.$$

Here $F = (z + f, w + g)$ is a holomorphic map with $f = O(|w| + |z|^2),\ g(z, w) = g(w) = O(w^2)$ and $u' = u + g(u)$.

Proof of Lemma 4.2: Let Φ_1 be a biholomorphic map, which maps M into M^N_{nor} defined by

$$w = z\bar{z} + 2Re\{z^s + \sum_{k=1}^{N}\sum_{j=2}^{s-1} a_{ks+j} z^{ks+j}\} + o(|z|^{sN+s-1}),$$

and let Φ_2 be a biholomorphic map from M' to ${M'}^N_{nor}$ with ${M'}^N_{nor}$ defined by

$$w' = z'\bar{z}' + 2Re\{{z'}^s + \sum_{k=1}^{N}\sum_{j=2}^{s-1} {a'}_{ks+j} {z'}^{ks+j}\} + o(|z'|^{sN+s-1}).$$

Define $\Psi = \Phi_2 \circ F \circ \Phi_1^{-1}$. Here we assume $\Phi_1,\ \Phi_2$ satisfy the normalization as in Theorem 3.1 at the origin. Then $\Psi(M^N_{nor})$ approximates ${M'}^N_{nor}$ up to order $\widetilde{N}$.

By Theorem 2.2, we conclude that

$$a_{ks+j} = a'_{ks+j}\ for\ ks + j \le \widetilde{N} \text{ and } \Psi = Id + O(|(z, w)|^N), \text{ with } \widetilde{N} = Ns + s - 1.$$

In what follows, we write $A_j(u),\ A^*_j(u),\ A^{nor}_j(u),\ A^{*nor}_j(u)$ for those quantities, defined as in (4.6), corresponding to $M,\ M',\ M^N_{nor},\ {M'}^N_{nor}$, respectively. Write h^{nor}_j and h^{*nor}_j for those holomorphic functions, defined before, corresponding to M^N_{nor} and ${M'}^N_{nor}$, respectively. Then from the way these functions were constructed, we have

$$h^{nor}_j(\zeta) = h^{*nor}_j(\zeta) + O(|\zeta|^N)\ for\ j = 1, 2, 3.$$

Hence,

$$A^{nor}_j(u) = A^{*nor}_j(u + \widetilde{g}(u)) + O(u^N),$$

where $\Psi = (z + \widetilde{f}(z), w + \widetilde{g}(w))$. This immediately gives the following:

$$F(A_j(u), u) = (A^*_j(u'), u') + O(u^N),$$

where $u' = u + g(u), F = (z + f(z, w), w + g(w))$. ∎

Summarizing the above, we have the following:

Proposition 4.3: *(1). Suppose that there is a holomorphic map $F : M \longrightarrow M'$ with $F = (z,w)+(O(|w|+|z|^2), O(w^2))$ such that $F(M)$ approximates M' up to order $\widetilde{N} = Ns+s-1 > s$ at 0. Then*

$$A_j^*(u+g(u)) = A_j(u) + O(u^N), \quad j = 0, 1, \cdots, s-1, \ u > 0.$$

(2). Suppose that there is a formal holomorphic map $F : M \longrightarrow M'$ with $F = (z,w) + (O(|w|+|z|^2), O(w^2))$. Then $A_j^*(u+g(u)) = f(A_j(u), u)$ in the formal sense. More precisely, let $f_{(\widetilde{N})}$, $g_{(\widetilde{N})}$ be the $(\widetilde{N}-1)^{th}$ truncation in the Taylor expansion of f and g, respectively. Then

$$A_j^*(u+g_{(\widetilde{N})}(u)) - f_{(\widetilde{N})}(A_j(u), u) = O(u^{N'})$$

where $N' \to \infty$ as $N \to \infty$. Indeed, we can choose $N' = N$.

Let $z = r\sigma(\tau, r)$ with $u = r^2$ be the conformal map from the disk $r\Delta := \{\tau \in \mathbb{C} : \ |\tau| < r\}$ to $D(u)$ with $\sigma(0,r) = 0, \ \sigma'_\tau(0,r) > 0$. Here, as defined before,

$$D(u) = \{z \in \mathbb{C}^1 : z\bar{z} + z^s + \bar{z}^s + E(z,\bar{z}) < u = r^2\}.$$

Similarly, let $z = r\sigma^*(\tau^*, r)$ with $u = r^2$ be the conformal map from the disk $r\Delta$ to $D^*(u)$ with $\sigma^*(0,r) = 0, \ {\sigma^*}'_\tau(0,r) > 0$. Here,

$$D^*(u) = \{z \in \mathbb{C}^1 : z\bar{z} + z^s + \bar{z}^s + E^*(z,\bar{z}) < u = r^2\}.$$

Then we know that $\sigma(\tau, r) = \tau(1+O(r))$ and σ is real analytic in (τ, r) over $\Delta_{1+\varepsilon} \times (-\varepsilon, \varepsilon)$ with $0 < \varepsilon \ll 1$. (See [Hu3]). Similar property also holds for σ^*.

Let $\tau_j(u) \in \Delta$ be such that $r\sigma(\tau_j(u), r) = A_j(u)$. Then

$$\tau_j(u) = \sigma^{-1}(\frac{A_j(u)}{u^{\frac{1}{2}}}, \sqrt{u}) = \frac{A_j(u)}{u^{\frac{1}{2}}}(1+O(\sqrt{u}))$$

Notice that $\frac{A_j(u)}{u^{\frac{1}{2}}} = \sum_{l=s-2}^{\infty} C_{l,j} u^{\frac{l}{2s}}$. Namely, $\frac{A_j(u)}{u^{\frac{1}{2}}}$ is analytic in $u^{\frac{1}{2s}}$. Here

$$C_{s-2,j} = s(s-1)^{\frac{1-s}{s}} e^{\frac{\pi\sqrt{-1}(1+2j)}{s}}. \tag{4.7}$$

Now, the hyperbolic distance between $A_1(u)$ and $A_2(u)$ as points in $D(u)$ is the same as the one between τ_1 and τ_2 as points in Δ. Let $L_{1(j+1)}(u) = e^{d_{hyp}(\tau_0, \tau_j)} - 1$. In particular, $L_{12}(u) = e^{d_{hyp}(\tau_0,\tau_1)} - 1$ Then since

$$d_{hyp}(\tau_0, \tau_1) = \frac{1}{2}\ln\left(\frac{1+|\frac{\tau_0-\tau_1}{1-\bar{\tau}_0\tau_1}|}{1-|\frac{\tau_0-\tau_1}{1-\bar{\tau}_0\tau_1}|}\right), \text{ and}$$

$$L_{12}(u) = s(s-1)^{\frac{1-s}{s}}|e^{\frac{\sqrt{-1}\pi}{s}} - e^{\frac{3\sqrt{-1}\pi}{s}}|u^{\frac{s-2}{2s}} + o(u^{\frac{s-2}{2s}}),$$

we see that $L_{12}(u)$ is analytic in $u^{\frac{1}{2s}}$.

Next, suppose $F : M \longrightarrow M'$ is a biholomorphic map with $F = (\widetilde{f}, \widetilde{g}) = (z, w) + (O(|w| + |z|^2), O(w^2))$. Then $\widetilde{f} = z + f$ is a conformal map from $D(u)$ to $D^*(u')$ with $u' = u + g(u)$. Hence the hyperbolic distance between $A_1(u)$ to $A_2(u)$ is the same as that of the hyperbolic distance from $A_1^*(u')$ to $A_2^*(u')$, for $F(A_j(u), u) = (A_j^*(u'), u')$.

Now, suppose that F is a biholomorphic map with $F = (\widetilde{f}, \widetilde{g}) = (z, w) + (O(|w| + |z|^2), O(w^2))$ such that $F(M)$ approximates M' at 0 up to order $\widetilde{N} = Ns + s - 1 > s$. As before, we can assume that M, M' are already normalized up to order $\widetilde{N}$. Then $F = Id + O(|z, w|^N)$, $M = \{w = z\bar{z} + 2Re\{\varphi_0(z)\} + o(|z|^{\widetilde{N}})\}$, $M' = \{w = z\bar{z} + 2Re\{\varphi_0(z)\} + o(|z|^{\widetilde{N}})\}$, where $\varphi_0(z) = z^s + o(z^s)$, $u' = u + g(u) = u + o(|u|^N)$ and $\varphi_0^{(sk+j)}(0) = 0$ for $j = 0, 1 \mod (s)$.

From the way σ and σ^* were constructed, we can show that (see [Lemma 2.1, Hu3]):

$$\sigma^*(\tau, u') - \sigma(\tau, u) = \tau O(u^N).$$

Indeed, this follows from the following more general result:

Lemma 4.4: Let $\sigma(\xi, r) = \xi \cdot (1 + O(r))$ and $\sigma^*(\xi, r) = \xi \cdot (1 + O(r))$ be the biholomorphic map from the unit disk Δ to

$$\begin{aligned} D(r) &:= \{\xi \in \mathbb{C}(\approx \overline{\Delta}) : \ |\xi|^2 + rF_1(r, \xi, \overline{\xi}) < 1\}, \\ D^*(r) &:= \{\xi \in \mathbb{C}(\approx \overline{\Delta}) : \ |\xi|^2 + rF_1(r, \xi, \overline{\xi}) + r^m F_2(r, \xi, \overline{\xi}) < 1\}, \end{aligned} \tag{4.8}$$

respectively. Here $F_j(r, \xi, \overline{\xi})$ are real-valued real analytic functions in a neighborhood of $\{0\} \times \overline{\Delta} \times \overline{\Delta}$. Then there is a constant C, depending only on F_j, such that

$$|\sigma^*(\xi, r) - \sigma(\xi, r)| \leq C|\xi| r^m, \quad \xi \in \overline{\Delta}.$$

Proof of Lemma 4.4: From the way σ and σ^* were constructed (see [Lemma 2.1, Hu3]), there are $U, U^* \in C^\omega(\partial\Delta \times (-\epsilon_0, \epsilon_0))$ with $0 < \epsilon_0 << 1$ such that

$$\sigma(\xi, r) = \xi\left(1 + U(\xi, r) + \mathcal{H}(U(\cdot, r))\right), \ \sigma^*(\xi, r) = \xi\left(1 + U^*(\xi, r) + \mathcal{H}(U^*(\cdot, r))\right), \ \xi \in \partial\Delta.$$

Here $\mathcal{H}$ is the standard Hilbert transform and U, U^* satisfy the following equations:

$$U = G_1(r, \xi, U, \mathcal{H}(U)), \quad U^* = G_1(r, \xi, U^*, \mathcal{H}(U^*)) + r^m G_2(r, \xi, U^*, \mathcal{H}(U^*)),$$

where $G_j(r, \xi, x, y)$ are real analytic in (r, ξ, x, y) with $G_j \lesssim |r| + |x|^2 + |y|^2$. Notice by the implicit function (see [Lemma 2.1, Hu3]), $\|U\|_{1/2}$, $\|U^*\|_{1/2} \leq C_1|r|$ with $\|\cdot\|_{1/2}$ the Hölder-$\frac{1}{2}$ norm. Next, we have

$$\begin{aligned} U^* - U &= \textstyle\int_0^1 \frac{\partial G_1}{\partial x}(r, \xi, \tau U^* + (1-\tau)U, \tau\mathcal{H}(U^*) + (1-\tau)\mathcal{H}(U))(U^* - U)\mathrm{d}\tau \\ &+ \textstyle\int_0^1 \frac{\partial G_1}{\partial y}(r, \xi, \tau U^* + (1-\tau)U, \tau\mathcal{H}(U^*) + (1-\tau)\mathcal{H}(U))(\mathcal{H}(U^*) - \mathcal{H}(U))\mathrm{d}\tau \\ &+ r^m G_2(r, \xi, U^*, \mathcal{H}(U^*)). \end{aligned} \tag{4.9}$$

By noticing that the Hilbert transform is bounded acting on the Hölder space, we easily conclude the result in the lemma by letting $|r| << 1$. ∎

Now, by Lemma 4.4, we see that $\tau_j^*(u') = \tau_j(u) + O(u^N)$. Therefore

$$L_{12}(u') - L_{12}(u) = O(u^N).$$

In particular, if $F : M \longrightarrow M'$ is a formal equivalence map with $F = (\widetilde{f}, \widetilde{g}) = (z, w) + (O(|w| + |z|^2), O(w^2))$. Then

$$L_{12}^*(u') = L_{12}(u) \text{ in the formal sense.} \tag{4.10}$$

We next prove the following:

Lemma 4.5: Let $F : M \longrightarrow M'$ be a formal equivalence map with $F = (\widetilde{f}, \widetilde{g}) = (z, w) + (O(|w| + |z|^2), O(w^2))$. Write $F = (\widetilde{f}, \widetilde{g}) = (z + f, w + g)$ as before. Then $\widetilde{g}$ is convergent.

Proof of Lemma 4.5: By (4.10), we have

$$L_{12}^*(\widetilde{g}(u)) = L_{12}(u) \text{ in the formal sense.}$$

Write $u = V^{2s}$ and $\widetilde{g}(u) = U^{2s}$. Then

$$L_{12}^*(U^{2s}) = L_{12}(V^{2s}).$$

Notice that $L_{12}^*(U^{2s})$ and $L_{12}(V^{2s})$ now are analytic in U and V, respectively. Moreover,

$$L_{12}^*(U^{2s}) = (\psi^*(U))^{s-2}, \ L_{12}(V^{2s}) = (\psi(V))^{s-2}$$

with ψ, ψ^* invertible holomorphic map of $(\mathbb{C}, 0)$ to itself, and with $\psi'(0) = \psi^{*\prime}(0)$. Hence, we get

$$\widetilde{g}(u) = ((\psi^{*-1} \circ \psi)(u^{\frac{1}{2s}}))^{2s}.$$

On the other hand, $(\psi^{*-1} \circ \psi(z)^{\frac{1}{2s}})^{2s}$ defines a multiple valued holomorphic function near the origin. By the Puiseux expansion, we get

$$(\psi^{*-1} \circ \psi(u^{\frac{1}{2s}}))^{2s} = \sum_{j=2s}^{\infty} c_j u^{\frac{j}{2s}}$$

However, $(\psi^{*-1} \circ \psi(u^{\frac{1}{2s}}))^{2s}$ also admits a formal power series expansion. We conclude that $c_j = 0$ if $2s$ does not divide j. This proves the convergence of $\widetilde{g}(u)$. ∎

We next prove the following theorem:

Theorem 4.6: *Let M, M' be real analytic Bishop surfaces near 0 defined by an equation of the form as in (4.1). Suppose $F = (\widetilde{f}, \widetilde{g}) : (M, 0) \longrightarrow (M', 0)$ is a formal equivalence map. Then F is biholomorphic near 0.*

Proof of Theorem 4.6: We can assume that $\widetilde{f} = z + f$ with $wt_{nor}(f) \geq 2$ and $\widetilde{g} = w + g$ with $wt_{nor}(g) \geq 4$. By Lemma 4.4 and by considering $F_0 \circ F$ instead of F, where $F_0(z, w) = (z, g^{-1}(w))$, we can assume, without loss of generality, that $\widetilde{g} = w$. We will prove the convergence of $\widetilde{f}$ by the hyperbolic geometry associated to the surface discussed above.

By Proposition 4.3 (2), we first notice that

$$\widetilde{f}(A_j(u), u) = A_j^*(u) \text{ in the formal sense.}$$

Namely, $\widetilde{f_{(N)}}(A_j(u), u) = A_j^*(u) + o(u^{N'})$ for any N. Here, $\widetilde{f_{(N)}}$ is the N^{th}-truncation in the Taylor expansion of $\widetilde{f}$ at 0; and N' depends only on N with $N' \to \infty$ as $N \to \infty$.

Write $\widetilde{M}$ and $\widetilde{M'}$ for the holomorphic hull of M and M', respectively. We next construct a holomorphic map from $\widetilde{M} \setminus M$ to $\widetilde{M'} \setminus M'$ as follows:

For $(z, u) \in D(u) \times \{u\}$, let $\tau(u) \in \Delta$ be such that $r\sigma(\tau(u), r) = z, \ u = r^2$. Let $\Psi(\cdot, r)$ be a biholomorphic map from Δ to itself such that $\Psi(\tau_j(u), r) = \tau_j^*(u)$ for $j = 0, 1$. Here, to see the existence of $\Psi(\cdot, r)$, it suffices for us to explain that $d_{hyp}(\tau_0(u), \tau_1(u)) = d_{hyp}(\tau_0^*(u), \tau_1^*(u))$. But, this readily follows from (4.10) and Lemma 4.4. Now, let

$$\Psi_1 = \frac{\tau - \tau_0(u)}{1 - \bar{\tau}_0(u)\tau}, \Psi_1^* = \frac{\tau - \tau_0^*(u)}{1 - \bar{\tau}_0^*(u)\tau}, \ \Theta(\tau, r) = e^{-i\theta(r) + i\theta^*(r)}\tau,$$

where

$$\begin{aligned} \theta(r) &= arg\{\tfrac{\tau_1(u) - \tau_0(u)}{1 - \overline{\tau_0}(u)\tau_1(u)} \tfrac{1}{u^{\frac{s-2}{2s}}}\} \\ \theta^*(r) &= arg\{\tfrac{\tau_1^*(u) - \tau_0^*(u)}{1 - \overline{\tau_0^*}(u)\tau_1^*(u)} \tfrac{1}{u^{\frac{s-2}{2s}}}\}. \end{aligned}$$

Then

$$\Psi(\tau, r) = \Psi_1^{*-1}(\tau, r) \circ \Theta(\tau, r) \circ \Psi_1(\tau, r). \tag{4.11}$$

$\Psi(\tau, r)$ is analytic in $(\tau, u^{\frac{1}{2s}}) \in \Delta_{1+\varepsilon_0} \times (-\varepsilon_0, \varepsilon_0)$. (See [Lemma 2.1, Hu3]).

We notice that when f is a priori known to be convergent, we then have, by the uniqueness property of the Möbius transformation, that

$$\widetilde{f}(r\sigma(\xi, r), r^2) = r\sigma^*(\Psi(\xi, r), r^2). \tag{4.12}$$

Consider the angle Θ_j ($j = 2, \cdots, s-1$) from the geodesic connecting τ_j to τ_0 to the geodesic connecting τ_j to τ_1. As a function of u (or r), we see, as in the definition of $\Psi(\xi, r)$, that

$$\Theta_j(u) = arg\{\frac{\tau_1(u) - \tau_j(u)}{\tau_0(u) - \tau_j(u)} \cdot \frac{1 - \overline{\tau_j(u)}\tau_0(u)}{1 - \overline{\tau_j(u)}\tau_1(u)}\} = arg\{\frac{C_{s-2,2} - C_{s-2,j}}{C_{s-2,1} - C_{s-2,j}}\} + O(u^{1/(2s)}).$$

We can similarly define Θ_j^* for M'. Then the same argument which we used to show that $L_{12}(u) = L_{12}^*(u)$ can be used to prove that

$$\Theta_j(u) \equiv \Theta_j^*(u), \ \text{ and } \ L_{1(j+1)}(u) = L_{1(j+1)}^*(u).$$

Now, we can use a Möbius transformation to map τ_j to the origin and τ_2 to a point in the positive real line. Then we easily see that Θ_j and $L_{1(j+1)}$ uniquely determine $\tau_j(u)$. As an immediate consequence of such a consideration, we conclude that

Lemma 4.7: $\Psi(\tau_j(u), r) = \tau_j^*(u)$ for $j = 0, \cdots, s-1$.

Now, for $(z, u) \in \widetilde{M} \setminus M$ close to the origin, we define

$$f^*(z, u) = \sqrt{u}\sigma^*(\Psi(\sigma^{-1}(\frac{z}{\sqrt{u}}, \sqrt{u}), \sqrt{u}), \sqrt{u})$$

Then $f^*(z, u)$ is analytic in $\widetilde{M} \setminus M$. We next prove the following:

Lemma 4.8: $\forall \alpha \geq 0, \frac{\partial^\alpha f^*}{\partial z^\alpha}(0, u) = \frac{\partial \widetilde{f}}{\partial z^\alpha}(0, u)$ in the formal sense. Namely, letting $\widetilde{f_{(N)}}$ be the polynomial consisting of terms of degree $\leq N$ in the Taylor expansion of $\widetilde{f}$ at 0, then $\exists N'(N) \to \infty$ as $N \to \infty$ such that

$$\frac{\partial^\alpha f^*}{\partial z^\alpha}(0, u) = \frac{\partial^\alpha \widetilde{f_{(N)}}}{\partial z^\alpha}(0, u) + o(u^{N'}).$$

Proof of Lemma 4.8: Let $S(u)$ be the hyperbolic polygon in $D(u)$ with vertices $A_j(u)(j = 0, 1, \cdots, s-1)$, whose sides consist of the geodesic segments connecting the vertices. Let $S^*(u)$ be the one corresponding to M'. We notice that for any points $P, Q \in \Delta$, then the geodesic segment connecting P to Q is

$$\gamma_{P,Q}(t) = \frac{t\frac{Q-P}{1-Q\bar{P}} + P}{1 + t\bar{P} \cdot \frac{Q-P}{1-Q\bar{P}}}, \qquad 0 \leq t \leq 1.$$

Hence, by the same argument used in the proof of Lemma 4.2 and by making use of the property that $\widetilde{f}$ formally maps vertices to the corresponding ones, we see that for any point $P \in \partial S(u)$, we have

$$f^*(P) = \widetilde{f_{(N)}}(P) + Error(P).$$

Here

$$|Error(P)| \leq Cu^{N'} \text{ with } N'(N) \to \infty \text{ as } N \to \infty$$

and C is a constant independent of P.

Now, by the Cauchy formula,

$$\frac{\partial^\alpha f^*}{\partial z^\alpha}(0, u) = \frac{\alpha!}{2\pi\sqrt{-1}} \int_{\partial S(u)} \frac{f^*(\zeta, u)}{\zeta^{\alpha+1}} d\zeta$$

and

$$\frac{\partial^\alpha \widetilde{f_{(N)}}}{\partial z^\alpha}(0,u) = \frac{\alpha!}{2\pi\sqrt{-1}}\int_{\partial S(u)} \frac{\widetilde{f_{(N)}}(\zeta,u)}{\zeta^{\alpha+1}}d\zeta$$

Notice that for $z \in \partial S(u), |z| \gtrsim u^{\frac{s-1}{s}}$, it thus follows that

$$|\frac{\partial^\alpha f^*}{\partial z^\alpha}(0,u) - \frac{\partial^\alpha \widetilde{f_{(N)}}}{\partial z^\alpha}(0,u)| \lesssim O(u^{N'-\frac{s-1}{s}\alpha}). \tag{4.13}$$

This completes the proof of Lemma 4.8. ■

We continue our proof of Theorem 4.6. We notice that
$(i) : \sigma^*(\zeta,\sqrt{u})$ is analytic in $(\zeta,\sqrt{u})$ near $(0,0)$,
$(ii) : \Psi(\tau,\sqrt{u})$ is analytic in τ and $u^{\frac{1}{2s}}$ near $(0,0)$ and,
$(iii) : \sigma^{-1}(\frac{z}{\sqrt{u}},\sqrt{u})$ is analytic in $(\frac{z}{\sqrt{u}},\sqrt{u})$ near $(0,0)$, too.
Write

$$\Psi(\tau,\sqrt{u}) = \sum_{\alpha,\beta=0}^{\infty} a_{\alpha\beta}\tau^\alpha u^{\frac{\beta}{2s}}$$

and

$$\Psi(\tau;Y_1) = \sum_{\alpha,\beta=0}^{\infty} a_{\alpha\beta}\tau^\alpha Y_1^\beta$$

Then

$$H(X,Y_1,Y_2) = Y_2\sigma^*(\Psi(\sigma^{-1}(X,Y_2);Y_1),Y_2)$$

is analytic in X, Y_1, Y_2 near 0. Write

$$H(X,Y_1,Y_2) = \sum_{\alpha,\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma}X^\alpha Y_1^\beta Y_2^\gamma \tag{4.14}$$

Then

$$f^*(z,u) = H(\frac{z}{\sqrt{u}}, u^{\frac{1}{2s}}, \sqrt{u}) = \sum_{\alpha,\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma}z^\alpha u^{\frac{\gamma-\alpha}{2}+\frac{\beta}{2s}}$$

Hence,

$$\frac{\partial^\alpha \widetilde{f}}{\partial z^\alpha}(0,u) = \sum_{\alpha,\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma}\alpha! u^{\frac{\gamma-\alpha}{2}+\frac{\beta}{2s}}$$

in the formal sense. It thus follows that if $b_{\alpha\beta\gamma} \neq 0, \frac{\gamma-\alpha}{2} + \frac{\beta}{2s} = \beta'$ is a non-negative integer. Hence $f^*(z,u) = \sum_{\alpha,\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma}z^\alpha u^{\beta'}$.
Now, $|b_{\alpha\beta\gamma}| \lesssim R^{|\alpha|+|\beta|+|\gamma|}$ with $R \gg 1$ by (4.14). We see that $|b_{\alpha\beta\gamma}| \lesssim R^{2s|\alpha|+2s\beta'} \lesssim (R^{2s})^{\alpha+\beta'}$.

This shows that $f^*(z,u)$ is holomorphic in (z,u) near 0; and thus $\widetilde{f}$ is convergent, too. The proof of Theorem 4.6 is finally completed. ∎

Proofs of Theorem 1.2 and Corollary 1.4: Theorem 1.2 and Theorem 4.6 have the same content. The proof of Corollary 1.4 follows from Theorem 1.1 and Theorem 1.2. ∎

As another application of Theorem 1.2, we have the following

Corollary 4.9: *Let $(M,0)$ be a real analytic elliptic Bishop surface with the Bishop invariant vanishing and the Moser invariant $s<\infty$ at 0. Then any element in $aut_0(M)$ is a holomorphic automorphism of $(M,0)$.*

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